SCHOOL OF MATH

ALGEBRA 1 · CLASS 19

# Solving Quadratics by Completing the Square



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# Recognizing Perfect-Square Trinomials

Completing the Square • Algebra 1

## 1. The patterns

$$(x+a)^2 = x^2 + 2ax + a^2$$

$$(x-a)^2 = x^2 - 2ax + a^2$$

## 2. Look for three clues

- **First term:** $x^2$
- **Last term:** $a$ perfect square
- **Middle term:** twice the product of $x$ and $a$

## 3. Two worked examples

**1** $x^2 + 6x + 9 = (x+3)^2$

$9 = 3^2$ and $6x = 2 \cdot x \cdot 3$

**2** $x^2 - 10x + 25 = (x-5)^2$

$25 = 5^2$ and $-10x = -2 \cdot x \cdot 5$

## Check the middle term!

$x^2 + 6x + 16$ is **NOT** a perfect-square trinomial. Since $16 = 4^2$, the middle term would need to be $8x$.

## 5. Try it: Rewrite as a square.

**A.** $x^2 + 8x + 16$ **B.** $x^2 - 12x + 36$

# Recognizing perfect-square trinomials

**Q1.** Is the trinomial a perfect square? If so, write it as a squared binomial.

$$x^2 + 18x + 81$$

**Q2.** Is the trinomial a perfect square? If so, write it as a squared binomial.

$$x^2 - 22x + 121$$

**Q3.** Is the trinomial a perfect square? Explain.

$$x^2 + 10x + 36$$

**Q4.** Is the trinomial a perfect square? Explain.

$$x^2 - 8x - 16$$

**Q5.** Find both values of b that make the trinomial a perfect square.

$$x^2 + bx + 144$$

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**Algebra 1**

Class 19, Item 2 of 8

# Finding the Missing Constant

## Completing the Square

### Take half. Then square.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

To complete $x^2 + bx + ___$, add $\left(\frac{b}{2}\right)^2$.

1. Find $b$, the coefficient of $x$.
2. Divide $b$ by 2.
3. Square the result.

### Example 1: Positive coefficient

$$x^2 + 10x + ___$$

$$10 \div 2 = 5$$

$$5^2 = 25$$

$$x^2 + 10x + 25 = (x + 5)^2$$

### Example 2: Negative coefficient

$$x^2 - 8x + ___$$

$$-8 \div 2 = -4$$

$$(-4)^2 = 16$$

$$x^2 - 8x + 16 = (x - 4)^2$$

**Tip:** The added constant is never negative: squaring gives a nonnegative result.

### Your turn: Find the missing constant.

**A.** $x^2 + 12x + ___$

**B.** $x^2 - 6x + ___$

**C.** $x^2 + 3x + ___$

**D.** $x^2 - 7x + ___$

# Finding the missing constant

**Q6.** Find the constant that completes the square. Then write the result as a squared binomial.

$$x^2 + 16x + _____$$

**Q7.** Find the constant that completes the square. Then write the result as a squared binomial.

$$x^2 - 14x + _____$$

**Q8.** Find the constant that completes the square. Then write the result as a squared binomial.

$$x^2 + 5x + _____$$

**Q9.** Find the constant that completes the square. Then write the result as a squared binomial.

$$x^2 - 9x + _____$$

**Q10.** Find the constant that completes the square. Then write the result as a squared binomial.

$$x^2 + \frac{2}{3}x + _____$$

# Rewriting as a Perfect Square

*From three terms to one squared binomial*

## Recognize the pattern

$$x^2 + 2ax + a^2 = (x + a)^2$$

$$x^2 - 2ax + a^2 = (x - a)^2$$

## Method

Take half the coefficient of $x$. Use that number inside the parentheses. Check that its square matches the constant.

## Example 1: A positive middle term

$$x^2 + 14x + 49$$

Half of $14$ is $7$; $7^2 = 49$.

$$x^2 + 14x + 49 = (x + 7)^2$$

## Example 2: A negative middle term

$$x^2 - 18x + 81$$

Half of $-18$ is $-9$; $(-9)^2 = 81$.

$$x^2 - 18x + 81 = (x - 9)^2$$

## Verify by expanding

$$\begin{aligned}(x - 9)^2 &= (x - 9)(x - 9) \\ &= x^2 - 9x - 9x + 81 \\ &= x^2 - 18x + 81\end{aligned}$$

## Your turn: Write each as a squared binomial.

A. $x^2 + 4x + 4$ _____

B. $x^2 - 16x + 64$ _____

C. $x^2 + 20x + 100$ _____

D. $x^2 - 2x + 1$ _____

*The sign inside the parentheses matches the sign of the middle term.*

# Rewriting expressions as perfect squares

**Q11.** Write the trinomial as a squared binomial. Verify by expanding.

$$x^2 + 24x + 144$$

**Q12.** Write the trinomial as a squared binomial. Verify by expanding.

$$x^2 - 26x + 169$$

**Q13.** Write the trinomial as a squared binomial. Verify by expanding.

$$x^2 + x + \frac{1}{4}$$

**Q14.** Write the trinomial as a squared binomial. Verify by expanding.

$$x^2 - 11x + \frac{121}{4}$$

**Q15.** Fill in both blanks so the equation is true.

$$x^2 - 30x + _____ = (x - _____)^2$$

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**Algebra 1**

Class 19, Item 4 of 8

# Solving by Completing the Square

## When the coefficient of $x^2$ is 1

**Method:** Move the constant → Add $\left(\frac{b}{2}\right)^2$ to BOTH sides → Factor → Take square roots

## Worked Example

$$x^2 + 6x - 7 = 0$$

**1** **Move the constant.**

$$x^2 + 6x = 7$$

**2** **Complete the square on both sides.**

Half of 6 is 3, and $3^2 = 9$.

$$x^2 + 6x + 9 = 7 + 9$$

**3** **Rewrite as a square.**

$$(x + 3)^2 = 16$$

**4** **Take BOTH square roots.**

$$x + 3 = \pm 4$$

**5** **Solve.**

$$x = -3 \pm 4$$

$$x = 1 \text{ or } x = -7$$

**Keep the equation balanced:** add the same number to both sides.

## Another example

$$x^2 - 4x - 12 = 0$$

$$x^2 - 4x = 12$$

$$x^2 - 4x + 4 = 12 + 4$$

$$(x - 2)^2 = 16$$

$$x - 2 = \pm 4$$

$$x = 6 \text{ or } x = -2$$

## Your turn:

Complete the square to solve.

**A.** $x^2 + 8x - 9 = 0$

**B.** $x^2 - 6x - 16 = 0$

# Solving quadratics by completing the square

**Q16.** Solve by completing the square.

$$x^2 + 10x + 16 = 0$$

**Q17.** Solve by completing the square.

$$x^2 - 8x + 12 = 0$$

**Q18.** Solve by completing the square.

$$x^2 + 2x - 24 = 0$$

**Q19.** Solve by completing the square.

$$x^2 - 12x + 20 = 0$$

**Q20.** Solve by completing the square.

$$x^2 + 14x = -33$$

# Odd Coefficients & Fractions

## *Completing the Square*

### *Half of an odd number is a fraction—and that is OK!*

$$b = 5 \rightarrow \frac{b}{2} = \frac{5}{2} \rightarrow \left(\frac{b}{2}\right)^2 = \frac{25}{4}$$

### Solve $x^2 + 5x = 6$

**1** Take half, then square.

$$\left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

**2** Add to BOTH sides.

$$x^2 + 5x + \frac{25}{4} = 6 + \frac{25}{4}$$

**3** Rewrite and simplify.

$$\left(x + \frac{5}{2}\right)^2 = \frac{49}{4}$$

**4** Take both square roots.

$$x + \frac{5}{2} = \pm\frac{7}{2}$$

**5** Solve.

$$x = -\frac{5}{2} \pm \frac{7}{2}$$

**$x = 1$ or $x = -6$**

### *Try a negative middle term*

$$x^2 - 3x = 10$$

$$x^2 - 3x + \frac{9}{4} = 10 + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{49}{4}$$

$$x - \frac{3}{2} = \pm\frac{7}{2}$$

**$x = 5$ or $x = -2$**



Use exact fractions. Write 6 as $\frac{24}{4}$ before adding $\frac{25}{4}$.

### *Your turn: Complete the square to solve.*

**A.** $x^2 + 7x = 8$

**B.** $x^2 - 5x = 14$

# Completing the square with odd coefficients and fractions

**Q21.** Solve by completing the square.

$$x^2 + 3x = 10$$

**Q22.** Solve by completing the square.

$$x^2 - 7x = -10$$

**Q23.** Solve by completing the square.

$$x^2 + 9x + 14 = 0$$

**Q24.** Solve by completing the square.

$$x^2 - x = 12$$

**Q25.** Solve by completing the square. Give exact answers.

$$x^2 + 5x - 2 = 0$$

# When the Leading Coefficient Is Not 1

*Divide first. Then complete the square.*

**Rule** | Before using $\left(\frac{b}{2}\right)^2$, make the coefficient of $x^2$ equal to 1.

## Solve $2x^2 + 8x - 10 = 0$

1. Divide EVERY term by 2.

$$x^2 + 4x - 5 = 0$$

2. Move the constant.

$$x^2 + 4x = 5$$

3. Add $\left(\frac{4}{2}\right)^2 = 4$ to both sides.

$$x^2 + 4x + 4 = 5 + 4$$

4. Rewrite as a square.

$$(x + 2)^2 = 9$$

5. Take square roots and solve.

$$x + 2 = \pm 3$$

$$x = -2 \pm 3$$

$$x = 1 \text{ or } x = -5$$

## Fractions can appear

$$2x^2 + 6x - 8 = 0$$

$$x^2 + 3x - 4 = 0$$

$$x^2 + 3x = 4$$

$$x^2 + 3x + \frac{9}{4} = 4 + \frac{9}{4}$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{25}{4}$$

$$x + \frac{3}{2} = \pm\frac{5}{2}$$

$$x = 1 \text{ or } x = -4$$

⚠ Divide every term—including the constant.

## Your turn: Divide first, then solve.

**A.** $3x^2 + 12x - 15 = 0$

**B.** $2x^2 - 10x + 8 = 0$

# When the leading coefficient is not 1: divide first

**Q26.** Divide first, then solve by completing the square.

$$3x^2 + 18x + 15 = 0$$

**Q27.** Divide first, then solve by completing the square.

$$4x^2 - 8x - 32 = 0$$

**Q28.** Divide first, then solve by completing the square.

$$2x^2 + 10x + 12 = 0$$

**Q29.** Divide first, then solve by completing the square.

$$4x^2 + 4x - 24 = 0$$

**Q30.** Divide first, then solve by completing the square.

$$2x^2 - 5x - 3 = 0$$

# Square Roots & Checking Solutions

*Remember ± • Simplify radicals • Check both answers*

**Rule**

If $u^2 = k$ and $k > 0$, then $u = \pm\sqrt{k}$.

*If k = 0: one real solution. If k < 0: no real solutions.*

## Solve $x^2 + 4x - 4 = 0$

$$x^2 + 4x = 4$$

$$x^2 + 4x + 4 = 4 + 4$$

$$(x + 2)^2 = 8$$

$$x + 2 = \pm\sqrt{8}$$

$$\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$$

$$x = -2 \pm 2\sqrt{2}$$

**Two exact solutions**

$x_1 = -2 + 2\sqrt{2}$ $\quad$ $x_2 = -2 - 2\sqrt{2}$

## Check BOTH in the original equation

Rewrite $x^2 + 4x - 4$ as $(x + 2)^2 - 8$.

**For $x = -2 + 2\sqrt{2}$:**

$$(2\sqrt{2})^2 - 8 = 8 - 8 = 0 \checkmark$$

**For $x = -2 - 2\sqrt{2}$:**

$$(-2\sqrt{2})^2 - 8 = 8 - 8 = 0 \checkmark$$

***Do not drop the ± sign. Keep radicals exact until the final answer.***

## Your turn: Solve and check both solutions.

**A.** $x^2 - 6x + 1 = 0$

**B.** $x^2 + 2x - 5 = 0$

# Square roots and checking solutions

**Q31.** Solve by completing the square. Give exact answers and check both solutions.

$$x^2 + 6x + 2 = 0$$

**Q32.** Solve by completing the square. Give exact answers and check both solutions.

$$x^2 - 10x + 13 = 0$$

**Q33.** Solve by completing the square. How many real solutions are there?

$$x^2 + 16x + 64 = 0$$

**Q34.** Solve by completing the square, or explain why there are no real solutions.

$$x^2 + 4x + 9 = 0$$

**Q35.** Solve exactly, then approximate each solution to the nearest hundredth.

$$x^2 - 2x - 11 = 0$$

# Mixed Practice & Word Problems

*Complete the square. Solve. Interpret.*

✓ Divide first if needed • ✓ Add the same amount to both sides • ✓ Use ± • ✓ Check your answers

## A garden problem

A rectangular garden is 4 feet longer than it is wide. Its area is 45 square feet. Find its dimensions.




$$x(x+4) = 45$$
$$x^2 + 4x = 45$$
$$x^2 + 4x + 4 = 45 + 4$$
$$(x+2)^2 = 49$$
$$x + 2 = \pm 7$$
$$x = 5 \text{ or } x = -9$$

*A width must be positive, so reject −9.*

***Width: 5 ft • Length: 9 ft***

## Mixed practice: Solve by completing the square.

**A.** $x^2 + 10x - 3 = 0$

**B.** $x^2 - 7x = 8$

**C.** $2x^2 + 12x - 14 = 0$

**D.** $x^2 + 6x + 13 = 0$

*For D, decide whether real solutions exist.*

## Your turn: Model and solve

A rectangle is 6 inches longer than it is wide. Its area is 40 square inches. Find its dimensions.

**Hint:** *Let x be the width. Start with* $x(x+6) = 40$.

# Mixed practice and word problems

**Q36.** Solve by completing the square. Give exact answers.

$$x^2 + 12x - 5 = 0$$

**Q37.** Solve by completing the square. Give exact answers.

$$3x^2 - 12x + 6 = 0$$

**Q38.** Decide whether real solutions exist. Solve if they do.

$$x^2 + 2x + 10 = 0$$

**Q39.** A rectangle is 8 meters longer than it is wide. Its area is 65 square meters. Find its dimensions.

**Q40.** Two positive numbers differ by 10, and their product is 56. Find the numbers.

# Homework review

**R1.** Which trinomial is a perfect square: $x^2 + 30x + 225$ or $x^2 + 15x + 225$? Write it as a squared binomial.

**R2.** Find the constant that completes the square. Then write the result as a squared binomial.

$$x^2 - 13x + _____$$

**R3.** Write the trinomial as a squared binomial. Verify by expanding.

$$x^2 - 28x + 196$$

**R4.** Solve by completing the square.

$$x^2 + 6x - 16 = 0$$

**R5.** Solve by completing the square.

$$x^2 + 11x + 24 = 0$$

# Homework review (continued)

**R6.** Divide first, then solve by completing the square.

$$3x^2 - 6x - 45 = 0$$

**R7.** Solve by completing the square. Give exact answers and check both solutions.

$$x^2 - 8x + 11 = 0$$

**R8.** Solve by completing the square. Give exact answers.

$$2x^2 + 8x + 3 = 0$$

# Homework word problems

**W1.** A rectangle is 2 inches longer than it is wide. Its area is 48 square inches. Find its dimensions.

**W2.** If each side of a square patio is increased by 3 feet, its area becomes 64 square feet. How long is each side of the original patio?

**W3.** The square of a number plus 6 times the number equals 40. Find all possible numbers.

**W4.** The product of two consecutive positive integers is 132. Find the integers.

**W5.** A rectangular classroom rug has an area of 91 square feet. Its length is 6 feet more than its width. Find its dimensions.

# Homework word problems (continued)

**W6.** A ball is kicked upward. Its height in feet after t seconds is $h = -16t^2 + 64t$. When is the ball 48 feet high?

**W7.** An 8-inch by 10-inch photo is surrounded by a frame of uniform width. The photo and frame together cover 120 square inches. How wide is the frame?

**W8.** One leg of a right triangle is 7 cm longer than the other leg. The hypotenuse is 13 cm. Find the legs.

**W9.** A bakery sells $100 - 5p$ cookie boxes per week when the price is p dollars. At what prices is weekly revenue \$480?

**W10.** A ball is thrown upward from 6 feet above the ground. Its height after t seconds is $h = -16t^2 + 32t + 6$. When does it hit the ground? Give an exact answer and round to the nearest hundredth.

# Homework word problems (continued)

**W11.** A rocket's height in feet after t seconds is $h = -16t^2 + 96t + 4$. Complete the square to write h in the form $h = -16(t - k)^2 + m$. What is the maximum height, and when is it reached?

**W12.** Challenge: A farmer uses 40 meters of fencing for three sides of a rectangular pen; a barn wall forms the fourth side. The pen's area is 150 square meters. Find every possible set of dimensions.