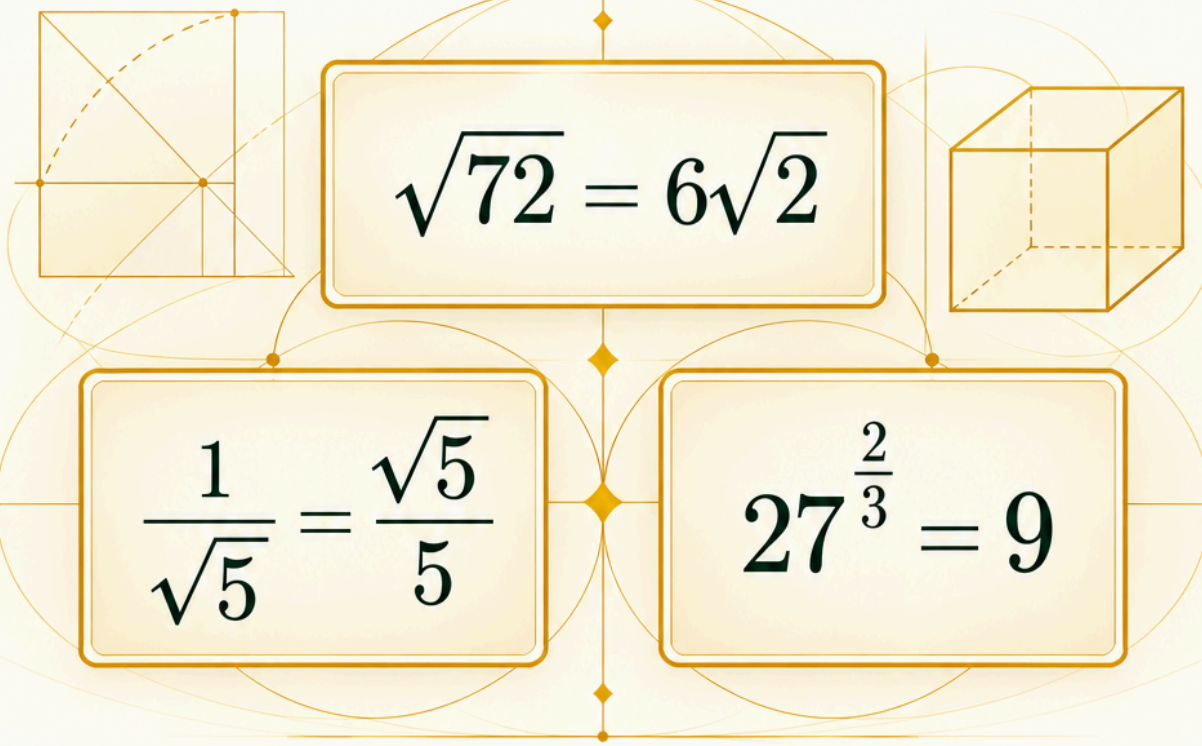
SCHOOL OF MATH

ALGEBRA 2

CLASS 15

Radical Expressions

Simplifying, Rationalizing, and Rational Exponents

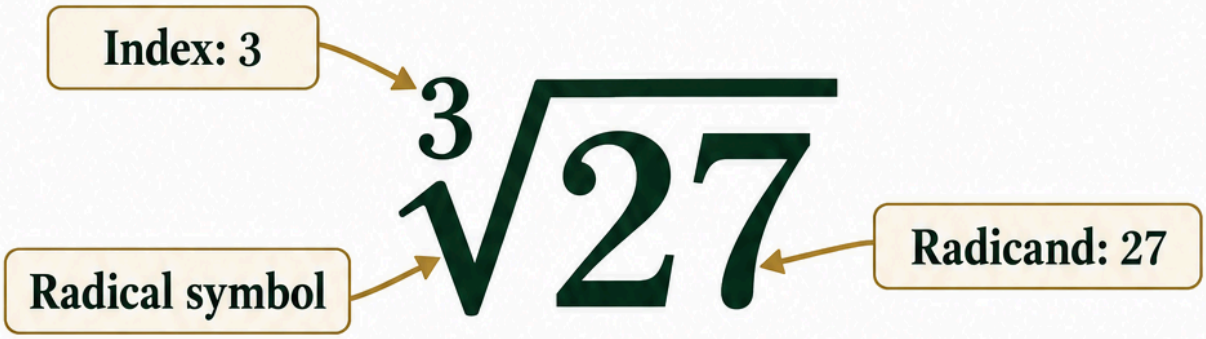


12 TOPICS • WORKED EXAMPLES • REAL-WORLD APPLICATIONS

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1. Understanding Radicals

The radical symbol, index, and radicand



RADICAL SYMBOL	INDEX	RADICAND
The root sign tells us to find a root.	The index tells us which root to find. If no index is shown, it is 2.	The number or expression inside the radical.

Read it. Understand it.

$$\sqrt{25} = 5$$

The square root of 25 is 5 because $5^2 = 25$.

$$\sqrt[3]{27} = 3$$

The cube root of 27 is 3 because $3^3 = 27$.

Remember: $\sqrt{25}$ means the principal square root, so its value is 5.

Understanding radicals

Q1. In $7\sqrt[3]{19}$, identify the coefficient, index, and radicand.

Q2. Find the principal fourth root $\sqrt[4]{81}$. Explain why the answer is not ± 3.

Q3. Which are real: $\sqrt{-49}$, $\sqrt[3]{-125}$, and $\sqrt{0}$? Evaluate those that are real.

Q4. Compare these two tasks: (a) evaluate $\sqrt{196}$; (b) solve $t^2 = 196$.

Q5. Find the real-number domain of $\sqrt{7 - 2x}$.

2. Square Roots and Cube Roots

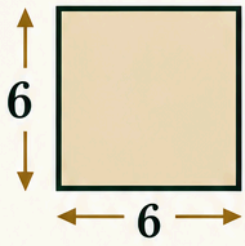
Recognizing perfect squares and perfect cubes

PERFECT SQUARES

A perfect square is an integer multiplied by itself.

$$6^2 = 6 \times 6 = 36$$

$$\sqrt{36} = 6$$



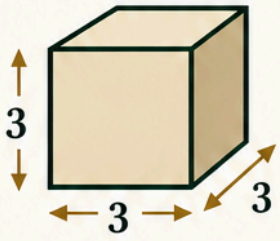
36 square units

PERFECT CUBES

A perfect cube is an integer multiplied by itself three times.

$$3^3 = 3 \times 3 \times 3 = 27$$

$$\sqrt[3]{27} = 3$$



27 cubic units

SQUARES TO RECOGNIZE

n	1	2	3	4	5	6	7	8	9	10
n^2	1	4	9	16	25	36	49	64	81	100

CUBES TO RECOGNIZE

n	1	2	3	4	5
n^3	1	8	27	64	125

Roots undo powers

$$\sqrt{81} = 9$$

The square root of 81 is 9 because $9^2 = 81$.

$$\sqrt[3]{64} = 4$$

The cube root of 64 is 4 because $4^3 = 64$.

$$\sqrt[3]{-8} = -2$$

The cube root of -8 is -2 because $(-2)^3 = -8$.

Remember: For real numbers: square roots of negative numbers are not real. Cube roots can be negative.

Square roots and cube roots

Q6. Evaluate $\sqrt{121}$.

Q7. Evaluate $\sqrt[3]{-216}$.

Q8. Evaluate $\sqrt{\frac{225}{64}}$.

Q9. Evaluate $\sqrt[3]{-\frac{343}{8}}$.

Q10. Evaluate $\sqrt{(-13)^2} - \sqrt[3]{-512}$.

3. Simplifying Radicals

Factor out perfect powers

Find a perfect-power factor that matches the root.

$$\sqrt{a^2 b} = |a|\sqrt{b} \quad (b \geq 0)$$

SQUARE ROOTS: LOOK FOR PERFECT SQUARES

$$\begin{aligned}\sqrt{72} &= \sqrt{36 \times 2} \\ &= \sqrt{36} \times \sqrt{2} \\ &= 6\sqrt{2}\end{aligned}$$

36 is a perfect square.

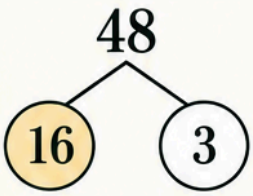
2 has no perfect-square factor greater than 1.

CUBE ROOTS: LOOK FOR PERFECT CUBES

$$\begin{aligned}\sqrt[3]{54} &= \sqrt[3]{27 \times 2} \\ &= \sqrt[3]{27} \times \sqrt[3]{2} \\ &= 3\sqrt[3]{2}\end{aligned}$$

$27 = 3^3$, so its cube root is 3.

Another way to see it



$$\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$$

Check: the remaining radicand should have no perfect-power factor greater than 1 that matches the index.

Simplifying radicals with perfect powers

Q11. Simplify $\sqrt{108}$.

Q12. Simplify $\sqrt{200}$.

Q13. Simplify $\sqrt[3]{375}$.

Q14. Simplify $3\sqrt{147}$.

Q15. Simplify both: (a) $\sqrt[3]{-432}$; (b) $\sqrt[4]{6480}$.

4. Radicals with Variables

Absolute values and domain restrictions

EVEN ROOTS NEED ABSOLUTE VALUES

$$\sqrt{x^2} = |x|$$

The principal square root is always nonnegative.

If $x = -5$:

$$\sqrt{(-5)^2} = \sqrt{25} = 5 = |-5|.$$

FACTOR OUT A PERFECT SQUARE

$$\sqrt{18x^2} = \sqrt{9 \cdot 2 \cdot x^2}$$
$$= 3|x|\sqrt{2}$$

Valid for every real x.

If $x \geq 0$, this becomes

$$3x\sqrt{2}.$$

ODD ROOTS KEEP THE SIGN

$$\sqrt[3]{x^3} = x$$

$$\sqrt[3]{16x^3} = \sqrt[3]{8 \cdot 2 \cdot x^3}$$
$$= 2x\sqrt[3]{2}$$

Valid for every real x; no absolute value is needed.

CHECK THE REAL-NUMBER DOMAIN

$\sqrt{x-4}$: $x - 4 \geq 0$, so $x \geq 4$.

$\dfrac{1}{\sqrt{x+2}}$ $x + 2 > 0$, so $x > -2$.

A square-root radicand must be nonnegative. A denominator must also be nonzero.

Remember: $\sqrt{x^2} = |x|$, while $\sqrt[3]{x^3} = x$.

Radicals with variables

Q16. Simplify $\sqrt{49x^2}$ for every real x.

Q17. Simplify $\sqrt{32x^6}$ for every real x.

Q18. Simplify $\sqrt[3]{-54x^4}$ for every real x.

Q19. Simplify $\sqrt{75a^4b^2}$ for real a and b.

Q20. Find the real domain of $\sqrt{\frac{x-1}{x+2}}$.

5. Adding and Subtracting Radicals

Simplify first, then combine like terms

WHAT MAKES RADICALS LIKE TERMS?

They have the same index and the same radicand after simplification.

$$3\sqrt{5} + 2\sqrt{5} = (3+2)\sqrt{5} = 5\sqrt{5}$$

ADDITION

$$\sqrt{8} + \sqrt{18}$$
$$= 2\sqrt{2} + 3\sqrt{2}$$
$$= 5\sqrt{2}$$

Simplify both radicals before combining.

SUBTRACTION

$$5\sqrt{12} - 2\sqrt{27}$$
$$= 10\sqrt{3} - 6\sqrt{3}$$
$$= 4\sqrt{3}$$

Subtract the coefficients; keep the radical.

UNLIKE RADICALS STAY SEPARATE

$$2\sqrt{3} + 4\sqrt{5}$$

Different radicands: these terms cannot be combined.

$$\sqrt{2} + \sqrt[3]{2}$$

Different indices: these terms cannot be combined.

Avoid this mistake: $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$, while $\sqrt{(9+16)} = 5$.

Adding and subtracting radicals

Q21. Simplify $7\sqrt{11} - 4\sqrt{11}$.

Q22. Simplify $\sqrt{48} + \sqrt{75}$.

Q23. Simplify $4\sqrt{20} - \sqrt{125}$.

Q24. Simplify $2\sqrt{45} + 3\sqrt{80} - \sqrt{5}$.

Q25. Simplify $\sqrt{12} + \sqrt{20} + \sqrt{27} - \sqrt{45}$.

6. Multiplying Radicals

Use the product rule and distribute.

THE PRODUCT RULE

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \quad (a \geq 0, b \geq 0)$$

For square roots, multiply the radicands, then simplify.

MULTIPLY AND SIMPLIFY

$$\sqrt{6} \cdot \sqrt{15} = \sqrt{90}$$
$$= \sqrt{9 \cdot 10}$$
$$= 3\sqrt{10}$$

Multiply the radicands, then simplify.

MULTIPLY COEFFICIENTS TOO

$$(2\sqrt{3})(4\sqrt{6})$$
$$= 8\sqrt{18}$$
$$= 24\sqrt{2}$$

Multiply the coefficients and the radical factors.

DISTRIBUTE TO EVERY TERM

$$\sqrt{3}(2 + \sqrt{12})$$
$$= 2\sqrt{3} + \sqrt{36}$$
$$= 2\sqrt{3} + 6$$

$$(\sqrt{2} + 3)(\sqrt{2} + 1)$$
$$= 2 + \sqrt{2} + 3\sqrt{2} + 3$$
$$= 5 + 4\sqrt{2}$$

Remember: addition requires like radicals; multiplication can combine different radicands with the same index.

Multiplying radicals

Q26. Simplify $\sqrt{14}\sqrt{21}$.

Q27. Simplify $(3\sqrt{5})(-2\sqrt{15})$.

Q28. Simplify $\sqrt{7}(2\sqrt{7}-\sqrt{28})$.

Q29. Expand and simplify $(\sqrt{6}+2)(\sqrt{6}-5)$.

Q30. Expand and simplify $(3\sqrt{2}-\sqrt{5})^2$.

7. Dividing Radicals

Use the quotient rule, then simplify.

THE QUOTIENT RULE

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \qquad (a \geq 0, b > 0)$$

Divide the radicands when the roots have the same index.

DIVIDE THE RADICANDS

$$\frac{\sqrt{72}}{\sqrt{8}} = \sqrt{\frac{72}{8}}$$
$$= \sqrt{9}$$
$$= 3$$

Divide the radicands, then simplify.

DIVIDE COEFFICIENTS TOO

$$\frac{6\sqrt{30}}{3\sqrt{5}} = 2\sqrt{\frac{30}{5}}$$
$$= 2\sqrt{6}$$

Divide the coefficients too, then simplify.

THE SAME IDEA WORKS FOR CUBE ROOTS

$$\frac{\sqrt[3]{54}}{\sqrt[3]{2}} = \sqrt[3]{\frac{54}{2}} = \sqrt[3]{27} = 3$$

Simplify a fraction inside a radical first: $\sqrt{\frac{12}{27}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$

Remember: A denominator can never be zero. For square roots in a denominator, the radicand must be positive.

Dividing radicals

Q31. Simplify $\frac{\sqrt{150}}{\sqrt{6}}$.

Q32. Simplify $\frac{10\sqrt{42}}{5\sqrt{7}}$.

Q33. Simplify $\frac{\sqrt[3]{-250}}{\sqrt[3]{2}}$.

Q34. Simplify $\sqrt{\frac{75}{108}}$.

Q35. Simplify $\frac{\sqrt{72x^4}}{\sqrt{2x^2}}$ for real x, and retain its restriction.

8. Rationalizing Monomial Denominators

Remove the radical from the denominator.

MULTIPLY BY A FORM OF 1

$$\frac{\sqrt{b}}{\sqrt{b}} = 1 \qquad (b > 0)$$

Multiply numerator and denominator by the same nonzero radical.

A SINGLE SQUARE ROOT

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$\sqrt{5} \cdot \sqrt{5} = 5.$

KEEP THE COEFFICIENT

$$\frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{6}$$

$2\sqrt{3} \cdot \sqrt{3} = 6.$

CUBE ROOTS: COMPLETE A PERFECT CUBE

$$\frac{1}{\sqrt[3]{2}} = \frac{1}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \frac{\sqrt[3]{4}}{2}$$

$\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{8} = 2.$

SIMPLIFY THE FINAL FRACTION

$$\frac{2}{\sqrt{8}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Remember: multiply both numerator and denominator.
The value stays the same because you multiply by 1.

Rationalizing monomial denominators

Q36. Rationalize $\frac{4}{\sqrt{7}}$.

Q37. Rationalize and reduce $\frac{9}{2\sqrt{6}}$.

Q38. Rationalize and simplify $\frac{\sqrt{10}}{\sqrt{15}}$.

Q39. Rationalize $\frac{5}{\sqrt[3]{9}}$.

Q40. Rationalize $\frac{2}{x\sqrt{3x}}$ and state the real-number domain.

9. Rationalizing Binomial Denominators

Multiply by the conjugate.

WHAT IS A CONJUGATE?

$$a + \sqrt{b} \quad \longleftrightarrow \quad a - \sqrt{b}$$

Keep the same two terms and change the sign between them.

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

The middle terms cancel: a difference of squares.

EXAMPLE 1: A NUMBER AND A RADICAL

$$\frac{1}{3 + \sqrt{2}} = \frac{1}{3 + \sqrt{2}} \cdot \frac{3 - \sqrt{2}}{3 - \sqrt{2}}$$

$$= \frac{3 - \sqrt{2}}{9 - 2} \qquad = \frac{3 - \sqrt{2}}{7}$$

The denominator becomes $3^2 - (\sqrt{2})^2 = 7$.

EXAMPLE 2: TWO RADICALS

$$\frac{2}{\sqrt{5} - \sqrt{3}} = \frac{2}{\sqrt{5} - \sqrt{3}} \cdot \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}}$$

$$= \frac{2(\sqrt{5} + \sqrt{3})}{5 - 3} \qquad = \sqrt{5} + \sqrt{3}$$

Use the conjugate $\sqrt{5} + \sqrt{3}$.

Remember: Multiply the entire numerator and denominator by the conjugate. Check that every denominator is nonzero.

Rationalizing binomial denominators

Q41. Rationalize $\frac{1}{4+\sqrt{3}}$.

Q42. Rationalize $\frac{3}{\sqrt{7}-2}$.

Q43. Rationalize $\frac{4}{\sqrt{11}+\sqrt{3}}$.

Q44. Rationalize and simplify $\frac{\sqrt{5}+1}{\sqrt{5}-1}$.

Q45. For $f(x)=\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}$, state the original domain, rationalize where possible, and find $f(2)$.

10. Rational Exponents

Connect radicals and exponent rules.

A FRACTIONAL EXPONENT REPRESENTS A ROOT

Denominator n: the root.

Numerator m: the power.

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

For $a > 0$ and positive integers m and n, with $n \geq 2$.

CONVERT BETWEEN FORMS

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\sqrt[3]{x^2} = x^{\frac{2}{3}}$$

$$\sqrt[4]{x^3} = x^{\frac{3}{4}}$$

Assume $x > 0$.

EVALUATE STEP BY STEP

$$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$$

$$16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$$

USE THE EXPONENT RULES

Assume $x > 0$.

Multiply: $x^{\frac{1}{2}} \cdot x^{\frac{3}{2}} = x^2$

Divide: $\dfrac{x^{5/3}}{x^{2/3}} = x$

Power of a power: $\left(x^{\frac{2}{3}}\right)^3 = x^2$

NEGATIVE EXPONENTS MEAN RECIPROCALS

$$8^{-\frac{2}{3}} = \frac{1}{8^{2/3}} = \frac{1}{4}$$

Remember: the denominator gives the root, the numerator gives the power, and a negative exponent gives a reciprocal.

Rational exponents

Q46. Evaluate $125^{2/3}$.

Q47. Evaluate $81^{3/4}$.

Q48. Evaluate $32^{-2/5}$.

Q49. For $x > 0$, simplify $\frac{x^{5/6}x^{1/3}}{x^{1/2}}$. Give exponent and radical forms.

Q50. For $x > 0$ and $y > 0$, simplify $(16x^8y^4)^{3/4}$.

11. Solving Radical Equations

Isolate, raise to a power, and check.

1. Isolate the radical. → 2. Raise both sides to the matching power. → 3. Solve. → 4. Check in the original equation.

EXAMPLE 1: ISOLATE BEFORE SQUARING

$$\begin{aligned} \sqrt{x+5}+1 &= 4 && \text{Original equation.} \\ \sqrt{x+5} &= 3 && \text{Subtract 1 from both sides.} \\ x+5 &= 9 && \text{Square both sides.} \\ x &= 4 && \text{Subtract 5.} \end{aligned}$$

Check: $\sqrt{4+5}+1 = 3+1 = 4.$ ✓

EXAMPLE 2: AN EXTRANEOUS SOLUTION

$$\begin{aligned} \sqrt{x+2} &= x && \text{The original equation requires } x \geq 0. \\ x+2 &= x^2 && \text{Square both sides.} \\ x^2 - x - 2 &= 0 && \text{Rewrite in standard form.} \\ (x-2)(x+1) &= 0 && \text{Factor.} \end{aligned}$$

Candidates: $x = 2$ or $x = -1$.

$x = 2$: $\sqrt{2+2} = 2$ ✓

$x = -1$: $\sqrt{-1+2} = 1 \neq -1$ ✗

Solution: $x = 2$ only.

CUBE ROOTS: CUBE BOTH SIDES

$$\sqrt[3]{x-1} = 2 \;\Rightarrow\; x-1 = 8 \;\Rightarrow\; x = 9.$$

Check: $\sqrt[3]{9-1} = 2.$

Remember: Squaring can introduce extraneous solutions. Always check each candidate in the original equation.

Solving radical equations

Q51. Solve $\sqrt{x+8} = 5$.

Q52. Solve $3\sqrt{2x-1} + 2 = 17$.

Q53. Solve $\sqrt{x+10} = x - 2$, rejecting any extraneous candidate.

Q54. Solve $\sqrt{3x+4} = x$. Show why one squared-equation candidate fails.

Q55. Solve $\sqrt{x+6} = \sqrt{x+1} + 1$.

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12. Real-World Applications

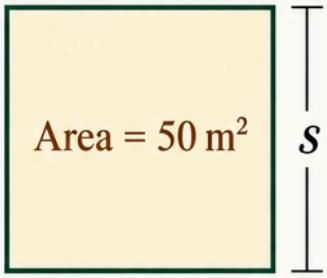
Radicals in geometry, distance, and measurement.

SQUARE ROOTS: FIND A SIDE FROM AREA

A square courtyard has an area of 50 m^2. What is its side length?

$$s = \sqrt{A}$$

$$s = \sqrt{50} = 5\sqrt{2} \approx 7.07 \text{ m}$$

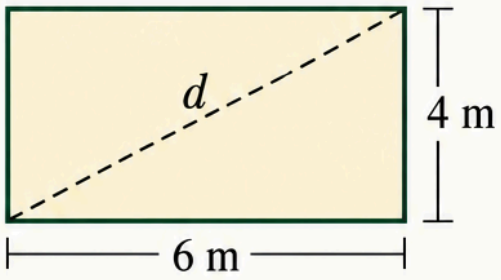


DISTANCE: FIND A RECTANGLE'S DIAGONAL

A rectangular garden is 6 m long and 4 m wide. Find the diagonal.

$$d = \sqrt{6^2 + 4^2}$$

$$d = \sqrt{52} = 2\sqrt{13} \approx 7.21 \text{ m}$$

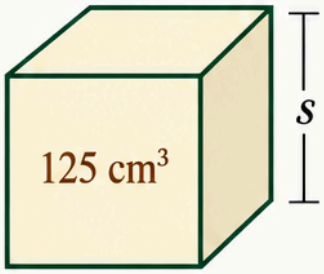


CUBE ROOTS: FIND AN EDGE FROM VOLUME

A cube-shaped container has a volume of 125 cm^3. Find its edge length.

$$s = \sqrt[3]{V}$$

$$s = \sqrt[3]{125} = 5 \text{ cm}$$



Remember: use square roots to undo squared lengths and cube roots to undo cubed lengths. Give your answer in the correct units.

Radicals in real-world applications

Q56. A square tile has area $98\ \text{cm}^2$. Find its exact side length.

Q57. A cube-shaped container has volume $250\ \text{cm}^3$. Find its exact edge length.

Q58. Find the exact distance between $A = (-2, 1)$ and $B = (4, 5)$, in coordinate units.

Q59. Use the supplied pendulum model $T = 2\pi\sqrt{L/9.8}$, with L in meters and T in seconds. Find the exact modeled period when $L = 2.45$.

Q60. A right triangle has legs x cm and $x + 2$ cm, with hypotenuse 10 cm. Model the hypotenuse with a radical and find both legs.

Homework review

R1. Explain the difference between $\sqrt{144}$ and the real solutions of $z^2 = 144$.

R2. Evaluate $\sqrt{\frac{361}{100}} + \sqrt[3]{-64}$.

R3. Simplify $2\sqrt{180} - \sqrt[3]{128}$.

R4. Simplify $\sqrt{48u^4v^2}$ for real u, v.

R5. Simplify $3\sqrt{28} - 2\sqrt{63} + \sqrt{175}$.

Homework review

R6. Expand and simplify $(\sqrt{3}+\sqrt{8})^2$.

R7. Simplify $\frac{12\sqrt{55}}{3\sqrt{5}}$.

R8. Rationalize $\frac{7}{\sqrt{28}}$.

R9. Rationalize $\frac{5}{3-\sqrt{2}}$.

R10. Evaluate $64^{-2/3}$.

Homework review

R11. Solve $\sqrt{2x+9} = x$ and check both candidates from squaring.

R12. A rectangle has side lengths $\sqrt{45}$ m and $\sqrt{20}$ m. Find its exact perimeter and area.

Homework word problems

W1. A square garden has area $162\ \text{m}^2$. Find its exact side length and the exact fencing length needed around all four sides.

W2. A cube-shaped gift box has volume $432\ \text{cm}^3$. Ribbon runs along four consecutive edges around one square face. Find the exact edge length and the ribbon length, with no extra ribbon for a knot.

W3. Two straight paths run due east and then due north, with lengths $\sqrt{75}$ m and $\sqrt{27}$ m. Find the total walking distance and the direct straight-line distance between start and finish.

W4. A rectangular panel has area $30\ \text{m}^2$ and length $3\sqrt{10}$ m. Find its exact width with a rationalized denominator, then its exact perimeter.

W5. A walker covers 18 km at a constant speed of $3+\sqrt{3}$ km/h. Find the exact travel time with a rational denominator, then round to the nearest hundredth of an hour.

Homework word problems

W6. A cube has volume $125\ \text{cm}^3$. A larger, similar cube has 8 times that volume. Using $S = 6V^{2/3}$, find both surface areas and their ratio. Explain why the ratio is not 8.

W7. A 13-foot ladder leans against a wall. Its foot is x feet from the wall and its top is $x + 7$ feet above the ground. Model the height as $\sqrt{169 - x^2}$ and find both distances.

W8. Two square murals have areas $x + 32$ and $x + 7$ square meters. The larger mural's side is exactly 1 meter longer. Find x, both areas, and both side lengths.