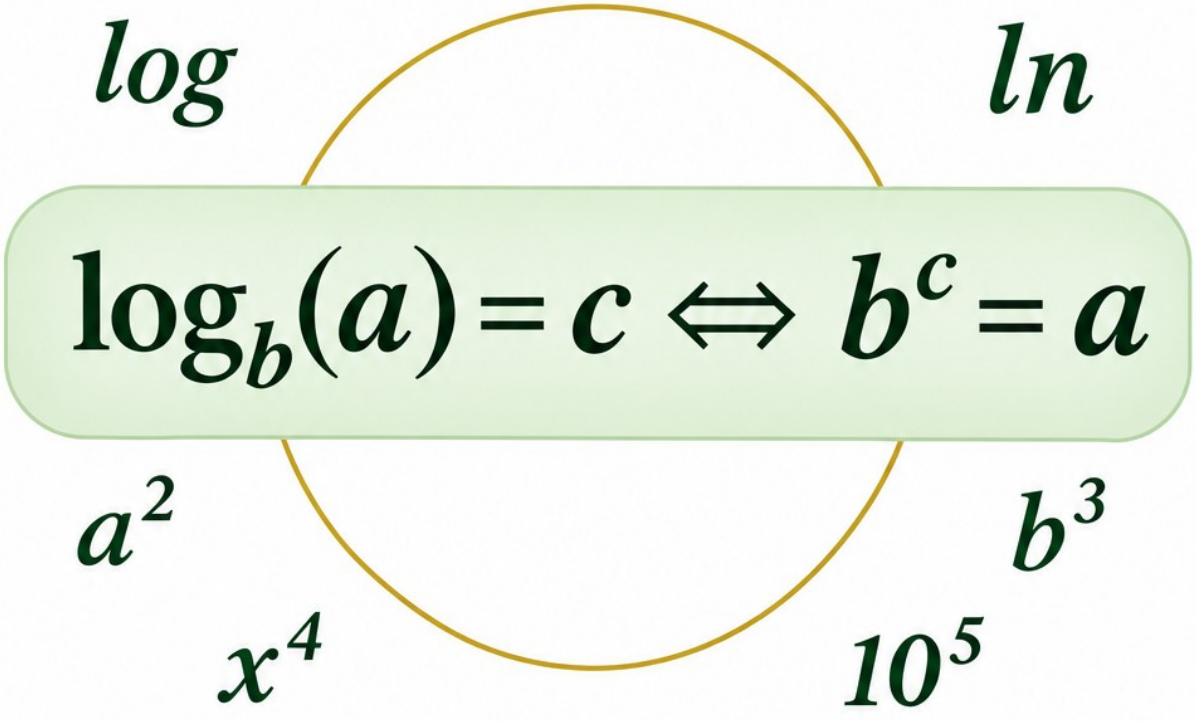
SCHOOL OF MATH

ALGEBRA 2 • CLASS 18

LOGARITHMS

Definition, Properties, and Change of Base



Understand the power behind the logarithm.

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What Is a Logarithm?

A logarithm answers: What exponent do we need?

$$\log_b(a) = c \iff b^c = a$$

b: base | a: argument | c: exponent

$$b > 0,\ b \neq 1,\ a > 0$$

Example: $\log_2(8) = 3$

because $2^3 = 8$

2 raised to what power equals 8? The answer is 3.

$\log_{10}(100) = 2$ | $\log_3(81) = 4$ | $\log_5(1) = 0$

A logarithm is an exponent.

What is a logarithm?

Q1. Evaluate the logarithm by asking which exponent is needed. State the question in words first.

$\log_3(243)$

Q2. Evaluate the logarithm. Then explain why the answer would be the same for any allowed base.

$\log_6(1)$

Q3. Evaluate the logarithm. Write the argument as a power of the base before answering.

$\log_{10}(10{,}000)$

Q4. Explain why neither expression represents a real number. Use the exponential form in your reasoning.

$\log_2(0) \quad \text{and} \quad \log_2(-8)$

Q5. Find the base b. Explain why only one value is acceptable.

$\log_b(49) = 2$

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Exponential and Logarithmic Forms

Two ways to express the same relationship.

$$b^c = a \iff \log_b(a) = c$$

b: base | c: exponent | a: result/argument

$b > 0,\ b \neq 1,\ a > 0$

Exponential → Logarithmic

$2^5 = 32$	→	$\log_2(32) = 5$
$10^{-2} = 0.01$	→	$\log_{10}(0.01) = -2$

Logarithmic → Exponential

$\log_3(81) = 4$	→	$3^4 = 81$
$\log_4(2) = \frac{1}{2}$	→	$4^{\frac{1}{2}} = 2$

Keep the base. The logarithm equals the exponent.

Try it: Rewrite $5^3 = 125$ in logarithmic form.

Answer: $\log_5(125) = 3$

Exponential and logarithmic forms

Q6. Rewrite the equation in logarithmic form. Identify the base, exponent, and argument.

$$6^3 = 216$$

Q7. Rewrite the equation in logarithmic form. Keep the sign of the exponent.

$$10^{-3} = 0.001$$

Q8. Rewrite the equation in exponential form.

$$\log_7(343) = 3$$

Q9. Rewrite the equation in exponential form. Then explain what the fractional exponent means.

$$\log_8(2) = \frac{1}{3}$$

Q10. Rewrite the first equation in exponential form and the second in logarithmic form. Treat the variables as the exponent or argument they represent.

$$\log_b(y) = x + 1 \qquad \text{and} \qquad 5^{2t} = w$$

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Evaluating Logarithms

Without a Calculator.

Ask: The base raised to what power gives the argument?

1 Positive powers

$\log_2(32) = 5$ *because* $2^5 = 32$

2 An argument of 1

$\log_7(1) = 0$ *because* $7^0 = 1$

3 Fractions and negative exponents

$\log_3\left(\frac{1}{27}\right) = -3$ *because* $3^{-3} = \frac{1}{27}$

4 Roots and fractional exponents

$\log_9(3) = \frac{1}{2}$ *because* $9^{\frac{1}{2}} = \sqrt{9} = 3$

Takeaway: Rewrite the argument as a power of the base.

Try it: $\log_5(125) \bullet \log_2\left(\frac{1}{8}\right) \bullet \log_{16}(4)$

Answers: $3 \bullet -3 \bullet \frac{1}{2}$

Evaluating logarithms without a calculator

Q11. Evaluate exactly.

$\log_4(64)$

Q12. Evaluate exactly. Show the negative exponent.

$\log_5\left(\frac{1}{125}\right)$

Q13. Evaluate exactly. Show the fractional exponent.

$\log_{27}(3)$

Q14. Evaluate exactly by writing the base and the argument as powers of the same number.

$\log_8(32)$

Q15. Evaluate exactly. The base is a fraction.

$\log_{1/2}(16)$

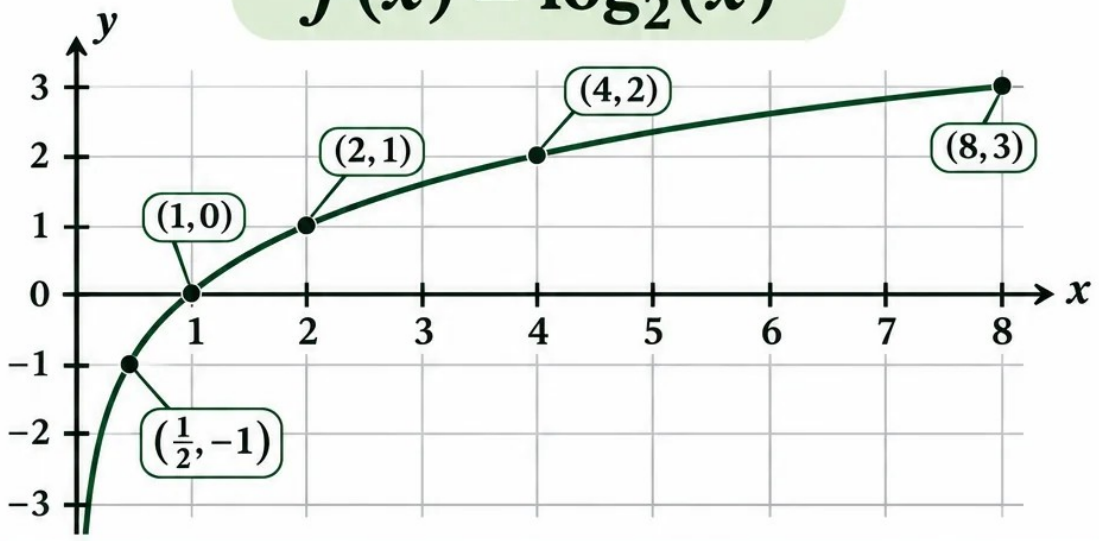
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Logarithmic Functions

Domain and Graphs

$$f(x) = \log_2(x)$$



Domain: $x > 0$

Range: all real numbers

Vertical asymptote: $x = 0$

x-intercept: $(1, 0)$

Inverse of an exponential function

$$y = \log_2(x) \iff x = 2^y$$

Exponential and logarithmic graphs reflect across $y = x$.

For $y = \log_b(x)$: increasing if $b > 1$; decreasing if $0 < b < 1$.
The argument of a real logarithm must be positive.

Logarithmic functions: domain and graphs

Q16. Make a table of values for x = 1/3, 1, 3, 9, and 27. Then state the x-intercept and the vertical asymptote, and sketch the graph.

$$f(x) = \log_3(x)$$

Q17. State the domain and the equation of the vertical asymptote.

$$g(x) = \log_2(x - 5)$$

Q18. State the domain in inequality form.

$$h(x) = \log(7 - 2x)$$

Q19. The graph of $y = 4^x$ passes through (3, 64). Name a point that must be on the graph of the inverse function, and explain the geometric relationship between the two graphs.

$$y = \log_4(x)$$

Q20. Compare the two functions: state whether each is increasing or decreasing, name a point they share, and explain how their outputs are related.

$$f(x) = \log_2(x) \qquad \text{and} \qquad g(x) = \log_{1/2}(x)$$

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Common and Natural Logarithms

Common Logarithm — Base 10

$$\log(x) = \log_{10}(x)$$

When no base is written, log usually means base 10.

$\log(1000) = 3$	because $10^3 = 1000$
$\log(0.01) = -2$	because $10^{-2} = 0.01$

Natural Logarithm — Base e

$$\ln(x) = \log_e(x)$$

$e \approx 2.71828$

$\ln(e^3) = 3$	The exponent of e is 3.
$\ln(1) = 0$	because $e^0 = 1$

Logarithms undo exponentials

$10^{\log x} = x$ $\quad$ $e^{\ln x} = x$ $\quad$ $x > 0$

Try it: $\log(100) \bullet \ln(e^5) \bullet \log(1)$

Answers: 2 • 5 • 0

log uses base 10. ln uses base e.

Both require a positive argument.

Common and natural logarithms

Q21. Evaluate both common logarithms exactly.

$$\log(100{,}000) \qquad \text{and} \qquad \log(0.0001)$$

Q22. Evaluate both natural logarithms exactly.

$$\ln(e^{7}) \qquad \text{and} \qquad \ln\left(\frac{1}{e^{2}}\right)$$

Q23. Simplify each expression without a calculator.

$$10^{\log 35} \qquad \text{and} \qquad e^{\ln 4.2}$$

Q24. Without a calculator, determine the two consecutive integers between which the value lies. Justify with powers of 10.

$$\log(4{,}500)$$

Q25. Rewrite the first equation in exponential form and the second in logarithmic form, then solve each for the variable.

$$\ln(x) = 5 \qquad \text{and} \qquad e^{2y} = 9$$

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Product and Quotient Properties

Product Property

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

A product inside becomes a sum outside.

$$\log_2(8 \times 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$$

Check: $\log_2(32) = 5$

Quotient Property

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

A quotient inside becomes a difference outside.

$$\log_3\left(\frac{81}{3}\right) = \log_3(81) - \log_3(3) = 4 - 1 = 3$$

Check: $\log_3(27) = 3$

Requirements:

$M > 0,\ N > 0,\ b > 0,\ b \neq 1$

These rules do not split sums!

$\log_b(M + N) \neq \log_b(M) + \log_b(N)$

Try it:

$$\log_5(25 \times 5) \quad \bullet \quad \log_2\left(\frac{64}{8}\right)$$

Answers: 3 • 3

Product and quotient properties

Q26. Evaluate using the product property, then confirm by multiplying first.

$$\log_3(27 \cdot 9)$$

Q27. Evaluate using the quotient property, then confirm by dividing first.

$$\log_2\left(\frac{128}{4}\right)$$

Q28. Use the given approximations to find each value to three decimal places. Do not use a calculator logarithm key.

$$\log_b(2) \approx 0.431, \quad \log_b(3) \approx 0.683; \qquad \text{find } \log_b(6) \text{ and } \log_b(1.5)$$

Q29. Write the expression as a single logarithm. State the restrictions on the variables.

$$\log_4(5) + \log_4(x) - \log_4(y)$$

Q30. Show by direct evaluation that the two expressions are not equal, and explain which property the mistake misuses.

$$\log_2(8+8) \quad \text{versus} \quad \log_2(8) + \log_2(8)$$

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The Power Property

Bring the exponent in front.

$$\log_b(M^p) = p\log_b(M)$$

$M > 0,\ b > 0,\ b \neq 1$

An exponent inside the logarithm becomes a coefficient outside.

Positive exponent

$$\log_2(8^2) = 2\log_2(8) = 2(3) = 6$$

Check: Since $8^2 = 64$, $\log_2(64) = 6$.

Negative exponent

$$\log_3(9^{-1}) = -\log_3(9) = -2$$

Check: Since $9^{-1} = \frac{1}{9}$, $\log_3\left(\frac{1}{9}\right) = -2$.

Fractional exponent

$$\log_4(\sqrt{16}) = \frac{1}{2}\log_4(16) = \frac{1}{2}(2) = 1$$

A square root is a power of $\frac{1}{2}$.

Use it in reverse: $3\log_5(x) = \log_5(x^3),\ x > 0$

Move the exponent, then evaluate the logarithm.

The power property

Q31. Evaluate using the power property, then confirm by computing the power first.

$\log_5\left(25^4\right)$

Q32. Evaluate using the power property.

$\log_2\left(16^{-1}\right)$

Q33. Evaluate exactly. Write the root as a fractional exponent first.

$\log_7\left(\sqrt[3]{49}\right)$

Q34. Use the power property in reverse to write each expression as a single logarithm with coefficient 1. Assume x > 0 and y > 0.

$5\log_2(x) \qquad \text{and} \qquad \frac{1}{2}\ln(y)$

Q35. Use the given approximation to find each value to three decimal places.

$\log_b(2) \approx 0.431; \qquad \text{find } \log_b(32) \text{ and } \log_b\left(\frac{1}{4}\right)$

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Expanding and Condensing

Logarithmic Expressions

Product → sum • Quotient → difference • Power → coefficient

EXPAND: One logarithm → several terms

$$\log_2\left(\frac{x^3y}{z^2}\right)$$
$$= \log_2(x^3) + \log_2(y) - \log_2(z^2)$$
$$= 3\log_2(x) + \log_2(y) - 2\log_2(z)$$

Assume $x > 0$, $y > 0$, $z > 0$.

CONDENSE: Several terms → one logarithm

$$2\ln(x) + \ln(y) - \ln(z)$$
$$= \ln(x^2) + \ln(y) - \ln(z)$$
$$= \ln\left(\frac{x^2y}{z}\right)$$

Keep the original restrictions: $x > 0, y > 0, z > 0$.

Check the domain first

$$\ln(x-2) + \ln(x+1) = \ln\left[(x-2)(x+1)\right]$$

Original domain: $x > 2$. Keep it after condensing.
Important: Keep the exact original domain, not $x < -1$.

Expand from the outside in. Condense by moving coefficients first.

Expanding and condensing logarithmic expressions

Q36. Expand completely. Assume all variables are positive.

$$\log_3\left(\frac{x^4 y}{z^5}\right)$$

Q37. Expand completely. Assume all variables are positive.

$$\ln\left(\frac{\sqrt{x}}{y^2}\right)$$

Q38. Expand completely and simplify any numerical logarithm. Assume x > 0.

$$\log(100x^3)$$

Q39. Condense into a single logarithm with coefficient 1. Assume all variables are positive.

$$3\log_2(x) + 2\log_2(y) - \log_2(z)$$

Q40. Condense into a single logarithm and state the domain of the original expression. Explain why the domain is not simply where the condensed argument is positive.

$$\log(x+3) + \log(x-3)$$

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Change of Base

Evaluate logarithms using log or ln.

$$\log_b(a) = \frac{\log(a)}{\log(b)} = \frac{\ln(a)}{\ln(b)}$$

$a > 0, b > 0, b \neq 1$

Argument on top. Base on bottom.

An exact value

$$\log_2(32) = \frac{\log(32)}{\log(2)} = 5$$

Check:
$2^5 = 32$

A calculator approximation

$$\log_3(20) = \frac{\ln(20)}{\ln(3)} \approx 2.727$$

Use parentheses:
$\ln(20) \div \ln(3)$.

Reasonableness check:
$3^2 < 20 < 3^3$, so
$2 < \log_3(20) < 3$.

Tip: Use the same logarithm in numerator and denominator.

$\frac{\log(a)}{\log(b)}$ ✓ | $\frac{\ln(a)}{\ln(b)}$ ✓

Try it: $\log_5(12) = \frac{\ln(12)}{\ln(5)} \approx 1.544$

Any valid new base gives the same answer.

Change of base

Q41. Use the change-of-base formula with common logarithms and simplify to an exact value without a calculator.

$\log_4(64)$

Q42. Approximate to three decimal places using natural logarithms. Then explain why the result is reasonable.

$\log_7(50)$

Q43. Approximate to three decimal places. State the two consecutive integers that trap the answer.

$\log_2(10)$

Q44. Approximate to three decimal places. Explain the sign of the answer before computing.

$\log_{0.5}(7)$

Q45. Rewrite the first quotient as a single logarithm, then evaluate the second quotient exactly.

$$\frac{\log_3(x)}{\log_3(5)} \quad \text{and} \quad \frac{\log(8)}{\log(2)}$$

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Solving Logarithmic and Exponential Equations

Rewrite • Solve • Check

1. Rewrite in exponential form

$$\log_2(x-1)=3$$

$$x-1=2^3$$

$$x=9$$

Domain: $x>1$. Check: $\log_2(8)=3$ ✓

2. Use logarithms to solve an exponential

$$3^x=20$$

$$x\ln(3)=\ln(20)$$

$$x=\frac{\ln(20)}{\ln(3)}\approx 2.727$$

3. Check for invalid solutions

$$\log_2(x)+\log_2(x-2)=3$$

Domain: $x>2$

$$\log_2[x(x-2)]=3$$

$$x^2-2x=8$$

$$(x-4)(x+2)=0$$

Candidates: $x=4$ or $x=-2$

Accept $x=4$. Reject $x=-2$ (outside the domain).

$$\log_2(4)+\log_2(2)=2+1=3$$ ✓

Every logarithm must have a positive argument.

Solving logarithmic and exponential equations

Q46. Solve. State the domain before solving and check the solution.

$$\log_3(x+4) = 2$$

Q47. Solve. Give the exact answer and a three-decimal approximation.

$$5^x = 40$$

Q48. Solve. State the domain, find all candidates, and reject any invalid values.

$$\log_2(x) + \log_2(x-6) = 4$$

Q49. Solve. Both sides are single logarithms with the same base.

$$\log_4(2x-1) = \log_4(x+5)$$

Q50. Solve. Isolate the power first. Give the exact answer and a three-decimal approximation.

$$2e^{3x} = 14$$

Homework review

R1. Evaluate exactly, and state the question the logarithm answers.

$\log_2(64)$

R2. Rewrite the first equation in logarithmic form and the second in exponential form.

$3^{-4} = \frac{1}{81}$ and $\log_{12}(144) = 2$

R3. Evaluate exactly by writing the base and the argument as powers of the same number.

$\log_{25}(125)$

R4. State the domain, the vertical asymptote, and the x-intercept.

$f(x) = \log_5(3x - 12)$

Homework review

R5. Evaluate exactly without a calculator.

$$\ln(e^{-3}) + \log(1{,}000)$$

R6. Use the given approximations to find each value to three decimal places.

$$\log_b(5) \approx 1.465, \quad \log_b(2) \approx 0.631; \qquad \text{find } \log_b(10) \text{ and } \log_b(2.5)$$

R7. Expand completely and simplify any numerical logarithm. Assume x > 0 and y > 0.

$$\log_5\left(\frac{25x^2}{\sqrt{y}}\right)$$

R8. Approximate to three decimal places using the change-of-base formula. Then explain why the result is reasonable.

$$\log_6(200)$$

Homework word problems

W1. The magnitude of an earthquake is M = log(I / I_0), where I_0 is a reference intensity. One earthquake has intensity 100,000 times I_0. Find its magnitude, and then determine how many times more intense a magnitude 7 earthquake is than a magnitude 5 earthquake.

$$M = \log\left(\frac{I}{I_0}\right)$$

W2. The pH of a solution is pH = −log[H^+], where [H^+] is the hydrogen-ion concentration in moles per liter. A sample of lemon juice has [H^+] = $10^{-2.4}$. Find its pH. Then find the hydrogen-ion concentration of a solution with pH 8.

$$\mathrm{pH} = -\log[\mathrm{H}^+]$$

W3. Sound level in decibels is L = 10 log(I / I_0), with I_0 = 10^{-12} watts per square meter. A vacuum cleaner produces intensity I = 10^{-5} watts per square meter. Find its sound level.

$$L = 10\log\left(\frac{I}{I_0}\right)$$

W4. A bacteria culture starts with 500 cells and doubles every hour, so N(t) = 500 · 2^t with t in hours. After how many hours does the culture reach 32,000 cells? Solve exactly.

$$N(t) = 500 \cdot 2^t$$

Homework word problems

W5. An account starts with $2,000 and earns 6% annual interest, compounded annually, with no deposits or withdrawals: A(t) = 2000(1.06)ᵗ. How many years does it take for the balance to reach $3,000? Give the answer to two decimal places, then state the first whole year in which the balance exceeds $3,000.

$$A(t) = 2000(1.06)^{t}$$

W6. A medication sample of 80 milligrams has a half-life of 10 days: m(t) = 80 · (1/2)^(t/10), with t in days. How long until 5 milligrams remain? Solve exactly.

$$m(t) = 80\left(\frac{1}{2}\right)^{t/10}$$

W7. A car bought for $28,000 loses 15% of its value each year: V(t) = 28000(0.85)ᵗ. After how many years is the car worth half its purchase price? Round to two decimal places and state the two consecutive whole years that bracket the answer.

$$V(t) = 28000(0.85)^{t}$$

W8. An investment of $5,000 grows with continuous compounding at 4.5% per year: A(t) = 5000e^(0.045t). How long does it take to reach $8,000? Round to two decimal places.

$$A(t) = 5000e^{0.045t}$$

Homework word problems

W9. A concert measures 90 decibels and a normal conversation measures 60 decibels. Using L = 10 log(I / I_0), find how many times more intense the concert sound is than the conversation.

$$L = 10\log\left(\frac{I}{I_0}\right)$$

W10. Solution A has pH 3 and Solution B has pH 5.5. Using pH = −log[H^+], find how many times greater the hydrogen-ion concentration of Solution A is than that of Solution B. Give an exact expression and a value rounded to the nearest whole number.

$$\mathrm{pH} = -\log[\mathrm{H}^+]$$

W11. A town of 42,000 people grows 2.3% per year: P(t) = 42000(1.023)^t. In how many years will the population reach 60,000? Round to one decimal place, then identify the first whole year in which the population exceeds 60,000 and show the two bracketing values.

$$P(t) = 42000(1.023)^t$$

W12. A cup of tea cools according to T(t) = 20 + 70e^(−0.08t), where T is the temperature in degrees Celsius and t is minutes after it is poured. Find its initial temperature, then find when the tea reaches 45°C. Round the time to one decimal place.

$$T(t) = 20 + 70e^{-0.08t}$$