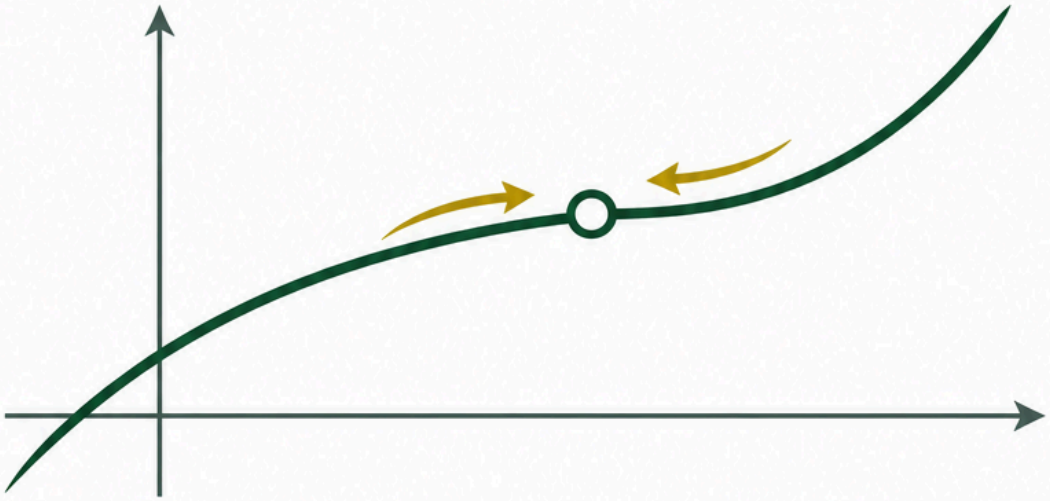
SCHOOL OF MATH

AP CALCULUS

CLASS 1

LIMITS

Graphical, Numerical, and Algebraic



$$\lim_{x \to a} f(x) = L$$

Explore how functions behave near a point.

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AP CALCULUS • CLASS 1 • ITEM 1

WHAT IS A LIMIT?

A limit describes the value $f(x)$ approaches as x gets close to a.

$$\lim_{x \to a} f(x) = L$$

Example: $f(x) = x + 1$

x	$f(x)$
1.9	2.9
1.99	2.99
2.01	3.01
2.1	3.1

As x approaches 2, $f(x)$ approaches 3.

$$\lim_{x \to 2} (x + 1) = 3$$

Think APPROACH, not necessarily REACH.

The limit depends on nearby values, not just the value at the point.

Q1

The table shows values of $f(x) = 2x - 1$ near $x = 3$. Use it to estimate $\lim_{x \to 3} f(x)$.

x	$f(x)$
2.9	4.8
2.99	4.98
3.01	5.02
3.1	5.2

Q2

Write the statement $\lim_{x \to 4} g(x) = 7$ in words, and explain what it says about the graph of g.

Q3

Complete the table for $f(x) = x^2$ near $x = 1$. Then estimate $\lim_{x \to 1} x^2$.

x	$f(x)$
0.9	
0.99	
1.01	
1.1	

Q4

Which statement must be true if $\lim_{x \to 5} h(x) = 2$?

A. $h(5)$ must equal 2.

B. As x gets close to 5, $h(x)$ gets close to 2.

C. $h(2) = 5$.

D. $h(x)$ never equals 2.

Q5

The function $f(x) = \dfrac{x^3 - 8}{x - 2}$ is undefined at $x = 2$. Use the table to estimate $\lim_{x \to 2} f(x)$.

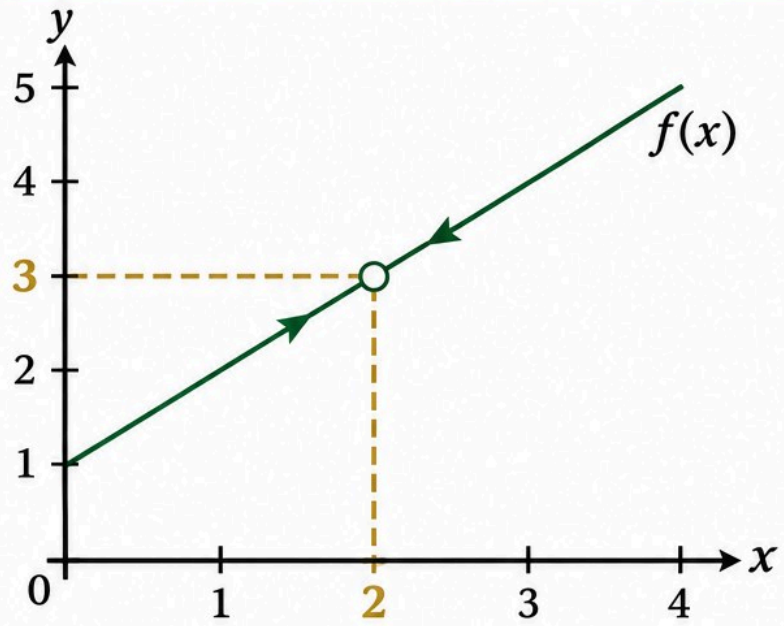
x	$f(x)$
1.9	11.41
1.99	11.9401
2.01	12.0601
2.1	12.61

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AP CALCULUS • CLASS 1 • ITEM 2

ESTIMATING LIMITS FROM GRAPHS

Follow the curve from *BOTH* sides of the x-value.



From the left: $f(x) \to 3$

From the right: $f(x) \to 3$

$$\lim_{x \to 2} f(x) = 3$$

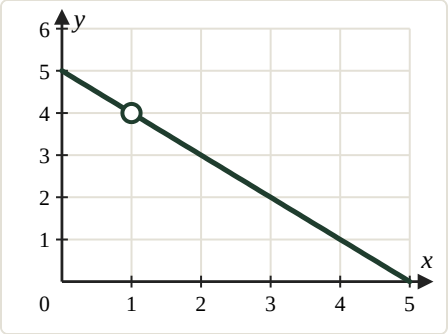
1. Locate $x = 2$.
2. Trace the graph toward $x = 2$ from each side.
3. Read the y-value both sides approach.

A hole does not prevent a limit from existing.

The graph approaches $y = 3$ even though $f(2)$ is undefined.

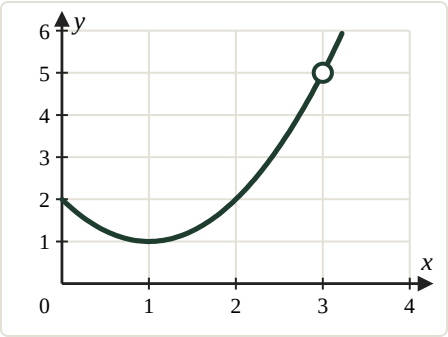
Q6

Use the graph to find $\lim_{x \to 1} f(x)$.



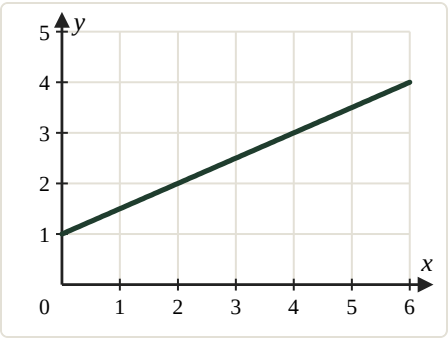
Q7

Use the graph of the parabola to find $\lim_{x \to 3} f(x)$.



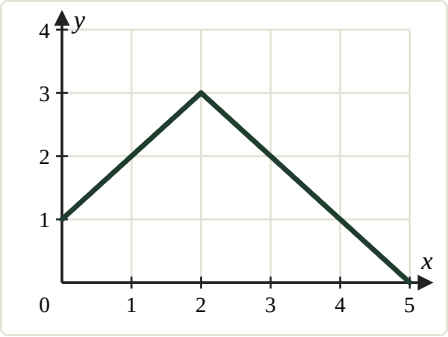
Q8

The graph shows $y = \frac{1}{2}x + 1$. Find $\lim_{x \to 4} f(x)$.



Q9

The graph has a corner at $x = 2$. Find (a) $\lim_{x \to 1} f(x)$, (b) $\lim_{x \to 2} f(x)$, (c) $\lim_{x \to 4} f(x)$.



Q10

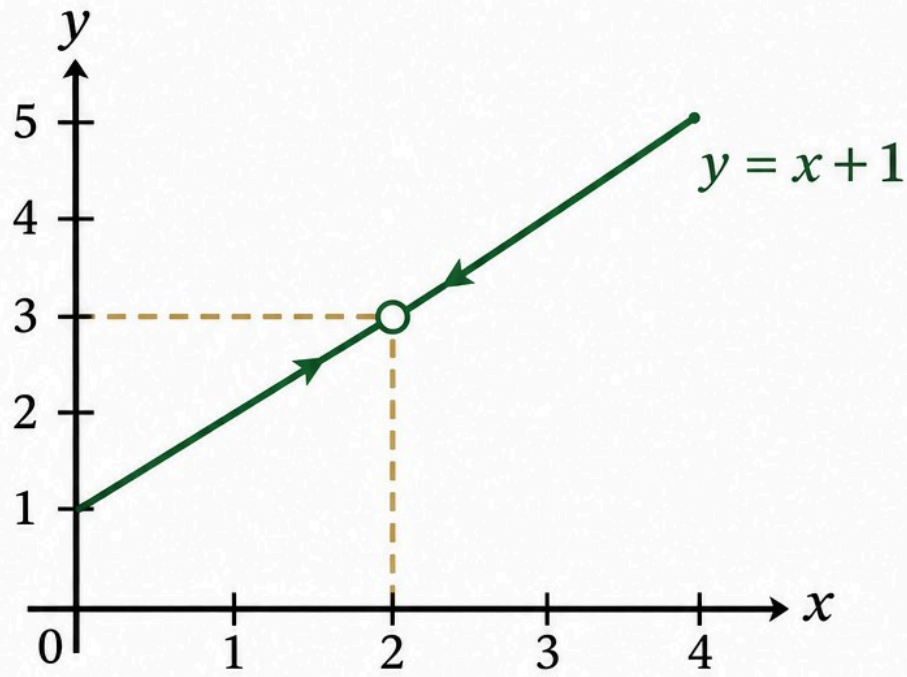
True or false: “If a graph has a hole at $x = a$, then $\lim_{x \to a} f(x)$ does not exist.” Explain.

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AP CALCULUS • CLASS 1 • ITEM 2

ESTIMATING LIMITS FROM GRAPHS

What y-value does the graph approach?



$x \to 2^-: f(x) \to 3$

$x \to 2^+: f(x) \to 3$

$$\lim_{x \to 2} f(x) = 3$$

Both sides approach the same y-value.

LOOK NEAR THE POINT

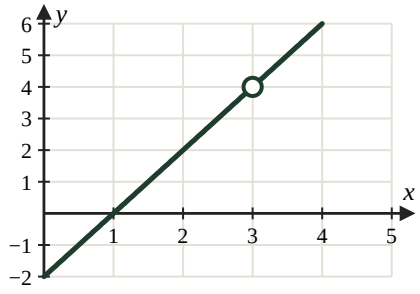
A hole at $x = 2$ does not stop the limit from being 3.

Q11

Write each sentence in limit notation: (a) As x approaches 5 from the left, $f(x)$ approaches 2. (b) As x approaches -1 from the right, $g(x)$ approaches 6.

Q12

Use the graph to complete: as $x \to 3^-$, $f(x) \to$ ___; as $x \to 3^+$, $f(x) \to$ ___; so $\lim_{x\to 3} f(x) =$ ___.



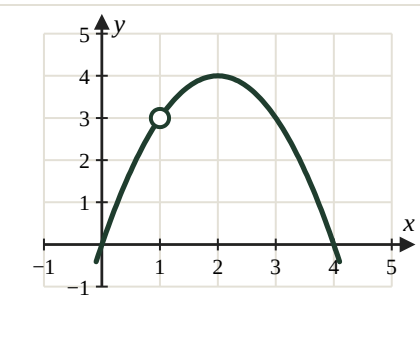
Q13

Use the table to estimate $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1} f(x)$.

x	$f(x)$
0.9	6.7
0.99	6.97
0.999	6.997
1.001	7.003
1.01	7.03
1.1	7.3

Q14

Use the graph to find $\lim_{x\to 1} f(x)$.



Q15

A student checks only $x = 2.1, 2.01, 2.001$ and concludes $\lim_{x\to 2} f(x) = 6$. What is missing from this reasoning?

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AP CALCULUS • CLASS 1 • ITEM 4

ONE-SIDED LIMITS

The direction of approach matters.

LEFT-HAND LIMIT

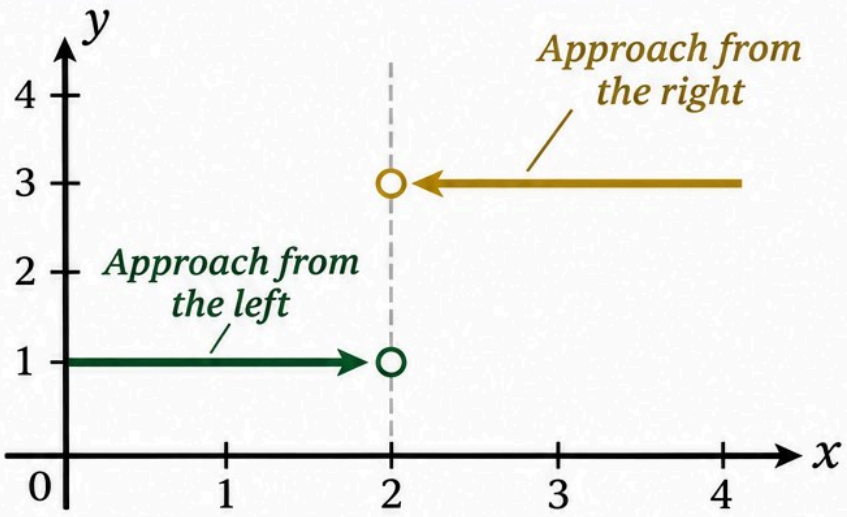
$$\lim_{x \to a^-} f(x)$$

Use x-values less than a.

RIGHT-HAND LIMIT

$$\lim_{x \to a^+} f(x)$$

Use x-values greater than a.



$$\lim_{x \to 2^-} f(x) = 1 \qquad \lim_{x \to 2^+} f(x) = 3$$

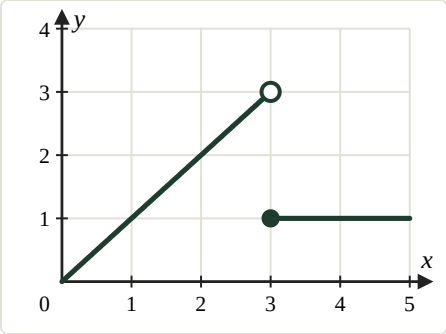
Different one-sided limits → no two-sided limit.

$$\lim_{x \to 2} f(x) \text{ does not exist.}$$

− means from the left. + means from the right.

Q16

Use the graph to find $\lim_{x\to 3^-} f(x)$, $\lim_{x\to 3^+} f(x)$, and $\lim_{x\to 3} f(x)$.



Q17

Let $f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 1, & x \geq 2 \end{cases}$. Find both one-sided limits at $x = 2$ and decide whether $\lim_{x\to 2} f(x)$ exists.

Q18

Let $g(x) = \dfrac{|x|}{x}$. Find $\lim_{x\to 0^-} g(x)$ and $\lim_{x\to 0^+} g(x)$.

Q19

Use the table to estimate both one-sided limits at $x = 2$. Does $\lim_{x\to 2} f(x)$ exist?

x	$f(x)$
1.9	2.1
1.99	2.01
1.999	2.001
2.001	5.002
2.01	5.02
2.1	5.2

Q20

The floor function $\lfloor x \rfloor$ gives the greatest integer less than or equal to x. Find $\lim_{x\to 2^-} \lfloor x \rfloor$ and $\lim_{x\to 2^+} \lfloor x \rfloor$.

WHEN DOES A LIMIT EXIST?

A finite two-sided limit exists when both one-sided limits equal the same real number.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

SAME VALUE → LIMIT EXISTS

$$\lim_{x \to 2^-} f(x) = 3 \quad | \quad \lim_{x \to 2^+} f(x) = 3$$

Therefore, $\lim_{x \to 2} f(x) = 3$.

DIFFERENT VALUES → LIMIT DOES NOT EXIST

$$\lim_{x \to 2^-} g(x) = 1 \quad | \quad \lim_{x \to 2^+} g(x) = 3$$

Therefore, $\lim_{x \to 2} g(x)$ does not exist.

CHECK BOTH SIDES

1. Find the left-hand limit.
2. Find the right-hand limit.
3. Compare their values.

The value at $x = a$ does not determine the limit.

Q21

Suppose $\lim\limits_{x\to 4^-} f(x) = 6$, $\lim\limits_{x\to 4^+} f(x) = 6$, and $f(4) = -1$. Find $\lim\limits_{x\to 4} f(x)$.

Q22

Let $h(x) = \begin{cases} x+4, & x < 1 \\ 0, & x = 1 \\ 2x+3, & x > 1 \end{cases}$. Find $\lim\limits_{x\to 1} h(x)$.

Q23

Find the value of k that makes $\lim\limits_{x\to 2} f(x)$ exist, where $f(x) = \begin{cases} kx+1, & x < 2 \\ x^2 - 1, & x \geq 2 \end{cases}$.

Q24

If $\lim\limits_{x\to a^-} f(x) = 4$ and $\lim\limits_{x\to a^+} f(x) = 4$, which statement must be true?

A. $f(a) = 4$

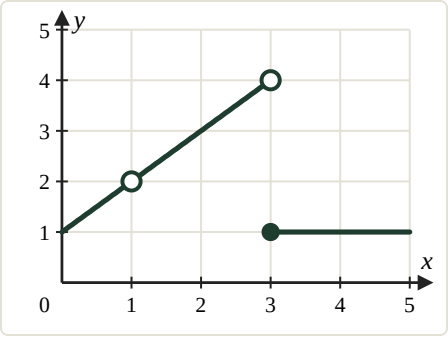
B. $\lim\limits_{x\to a} f(x) = 4$

C. $f(a)$ is undefined

D. $f(a) \neq 4$

Q25

Use the graph. Decide whether each limit exists, and give its value if it does: (a) $\lim\limits_{x\to 1} f(x)$ (b) $\lim\limits_{x\to 3} f(x)$ (c) $\lim\limits_{x\to 4} f(x)$.

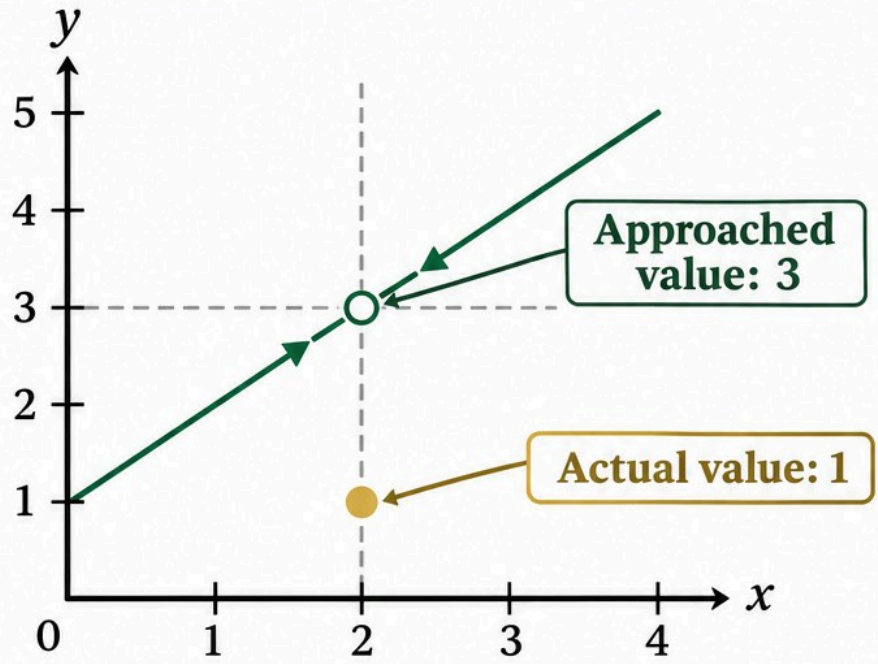


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AP CALCULUS • CLASS 1 • ITEM 6

THE LIMIT *vs.* THE FUNCTION VALUE

Approaching a point is different from evaluating at it.



THE LIMIT

$$\lim_{x \to 2} f(x) = 3$$

Where nearby outputs are heading.

FUNCTION VALUE

$$f(2) = 1$$

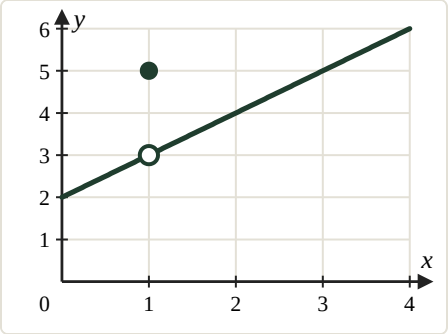
The output at $x = 2$.

A limit can exist even when it differs from $f(a)$.

A limit can also exist when $f(a)$ is undefined.

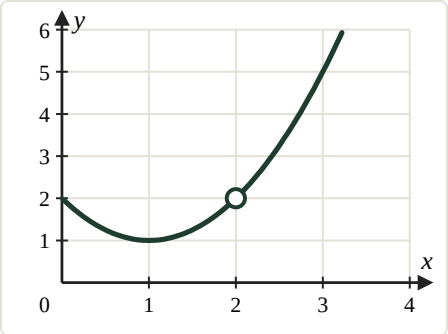
Q26

Use the graph to find $\lim_{x\to 1} f(x)$ and $f(1)$.



Q27

Use the graph to find $\lim_{x\to 2} f(x)$ and $f(2)$.



Q28

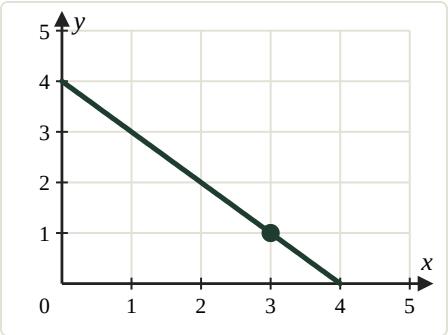
Let $g(x) = \begin{cases} 3x - 1, & x \neq 2 \\ 10, & x = 2 \end{cases}$. Find $\lim_{x\to 2} g(x)$ and $g(2)$.

Q29

True or false: "If $f(4) = 7$, then $\lim_{x\to 4} f(x) = 7$." Give a reason.

Q30

The graph shows $y = 4 - x$ with a filled dot at $(3, 1)$. Compare $\lim_{x\to 3} f(x)$ and $f(3)$.



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LIMIT LAWS & DIRECT SUBSTITUTION

If $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$:

SUM / DIFFERENCE	$\lim_{x\to a} [f(x) \pm g(x)] = L \pm M$
PRODUCT	$\lim_{x\to a} [f(x)g(x)] = LM$
QUOTIENT	$\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}, \quad M \neq 0$
POWER	$\lim_{x\to a} [f(x)]^n = L^n$ *n is a positive integer.*

DIRECT SUBSTITUTION

If a function is continuous at a, substitute $x = a$.

$$\lim_{x\to 2} (3x^2 - 4x + 1)$$
$$= 3(2)^2 - 4(2) + 1$$
$$= 12 - 8 + 1 = \boxed{5}$$

For rational functions, check that the denominator is not zero.

A $\frac{0}{0}$ *result needs another method.*

Q31

Given $\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = -2$, find $\lim_{x\to 3}[2f(x) - 3g(x)]$.

Q32

With the same limits as Q31, find (a) $\lim_{x\to 3}[f(x)g(x)]$, (b) $\lim_{x\to 3} \dfrac{f(x)}{g(x)}$, (c) $\lim_{x\to 3}[g(x)]^3$.

Q33

Evaluate $\lim_{x\to -1} (2x^3 + x^2 - 5)$.

Q34

Evaluate $\lim_{x\to 2} \dfrac{x^2 + 1}{x + 3}$.

Q35

Try direct substitution on each limit. State the result, or say that another method is needed: (a) $\lim_{x\to 3} \dfrac{x + 5}{x - 2}$ (b) $\lim_{x\to 4} \dfrac{x^2 - 16}{x - 4}$.

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AP CALCULUS • CLASS 1 • ITEM 8

LIMITS BY FACTORING

Simplify first. Then evaluate the limit.

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

1. TRY SUBSTITUTION

$$\frac{3^2 - 9}{3 - 3} = \frac{0}{0}$$

Indeterminate form — keep working.

2. FACTOR THE NUMERATOR

$$x^2 - 9 = (x - 3)(x + 3)$$

3. CANCEL THE COMMON FACTOR

$$\frac{\cancel{(x - 3)}(x + 3)}{\cancel{x - 3}}$$

$$= x + 3, \text{ for } x \neq 3$$

4. EVALUATE THE SIMPLIFIED LIMIT

$$\lim_{x \to 3} (x + 3) = 3 + 3 = \boxed{6}$$

Why is cancellation allowed?

A limit uses nearby x-values, where $x \neq 3$.

The original expression is still undefined at $x = 3$.

Q36

Evaluate $\lim_{x \to 5} \dfrac{x^2 - 25}{x - 5}$.

Q37

Evaluate $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$.

Q38

Evaluate $\lim_{x \to -3} \dfrac{x^2 + 5x + 6}{x + 3}$.

Q39

Evaluate $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

Q40

Evaluate $\lim_{x \to 4} \dfrac{x^2 - 16}{x^2 - 3x - 4}$.

LIMITS BY RATIONALIZING

Use the conjugate to remove a radical difference.

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$$

1 TRY SUBSTITUTION

$$\frac{\sqrt{4}-2}{0} \to \frac{0}{0}.$$ *Indeterminate form.*

2 MULTIPLY BY THE CONJUGATE

$$\frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

3 SIMPLIFY AND CANCEL

$$\frac{(x+4)-4}{x(\sqrt{x+4}+2)} = \frac{x}{x(\sqrt{x+4}+2)} = \frac{1}{\sqrt{x+4}+2} \quad \text{For } x \neq 0.$$

4 EVALUATE THE LIMIT

$$\lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

Conjugates use the difference of squares:

$$(A-B)(A+B) = A^2 - B^2$$

Q41

Evaluate $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$.

Q42

Evaluate $\displaystyle\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$.

Q43

Evaluate $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$.

Q44

The radical is in the denominator. Evaluate $\displaystyle\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}$.

Q45

Evaluate $\displaystyle\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x}$.

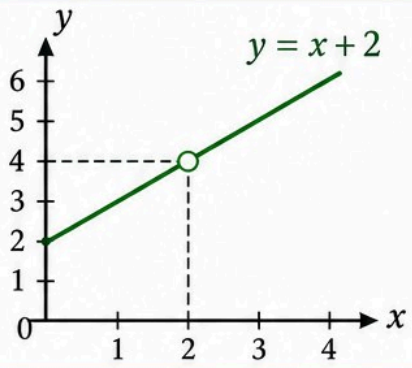
SCHOOL OF MATH

AP CALCULUS • CLASS 1 • ITEM 10

ONE LIMIT, THREE METHODS

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

1 GRAPHICAL



Both sides approach $y = 4$.

2 NUMERICAL

x	$f(x)$
1.9	3.9
1.99	3.99
2.01	4.01
2.1	4.1

Nearby values suggest a limit of 4.

3 ALGEBRAIC

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2. \quad \text{For } x \neq 2.$$

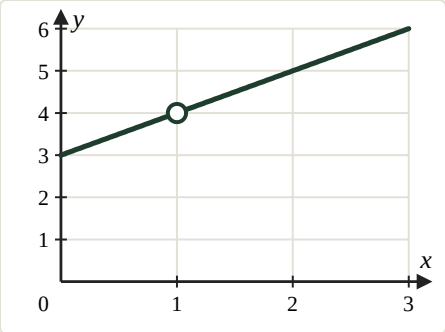
$$\lim_{x \to 2} (x + 2) = 4.$$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

Graphs and tables suggest the limit; algebra justifies the exact value.

Q46

Find $\lim_{x\to 1}\dfrac{x^2+2x-3}{x-1}$ three ways: (a) from the graph, (b) by completing the table, (c) algebraically.



x	$f(x)$
0.9	
0.99	
1.01	
1.1	

Q47

Complete the table for $f(x)=\dfrac{x^2-x-2}{x-2}$, estimate the limit as $x\to 2$, then confirm it algebraically.

x	$f(x)$
1.9	
1.99	
2.01	
2.1	

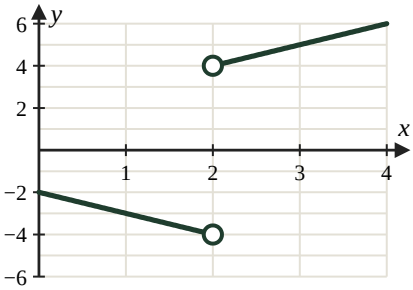
Q48

The table shows $f(x)=\dfrac{\sqrt{x+16}-4}{x}$ near 0 (rounded to five decimal places). Estimate the limit, then find its exact value.

x	$f(x)$
−0.1	0.12520
−0.01	0.12502
0.01	0.12498
0.1	0.12481

Q49

Let $f(x)=\dfrac{x^2-4}{|x-2|}$. Use the graph and algebra to find both one-sided limits at $x=2$. Does $\lim_{x\to 2} f(x)$ exist?



Q50

Name the best method for each limit, then evaluate it: (a) $\lim_{x\to 2}\dfrac{x^3-8}{x-2}$ (b) $\lim_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$ (c) $\lim_{x\to -1}(x^2+4x)$.

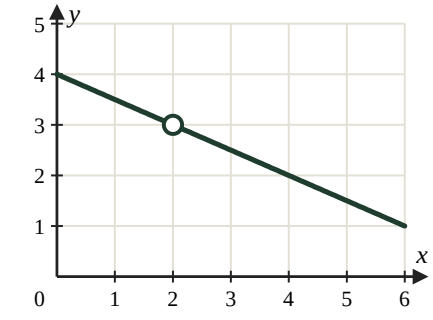
R1

Use the table for $f(x) = 3x + 2$ to estimate $\lim_{x \to 1} f(x)$.

x	$f(x)$
0.9	4.7
0.99	4.97
1.01	5.03
1.1	5.3

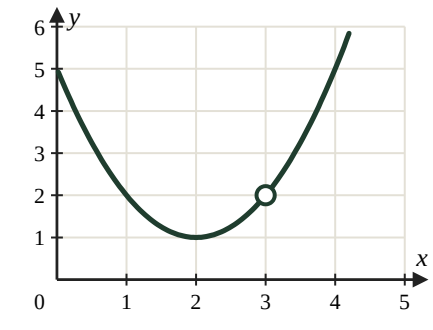
R2

Use the graph to find $\lim_{x \to 2} f(x)$.



R3

Use the graph to find $\lim_{x \to 3^-} f(x)$, $\lim_{x \to 3^+} f(x)$, and $\lim_{x \to 3} f(x)$.



R4

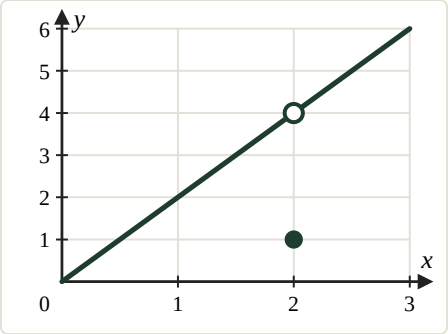
Let $f(x) = \begin{cases} x + 2, & x < 1 \\ 5 - x, & x \geq 1 \end{cases}$. Find both one-sided limits at $x = 1$ and decide whether the limit exists.

R5

Find c so that $\lim_{x \to 3} f(x)$ exists, where $f(x) = \begin{cases} x^2 + c, & x < 3 \\ 2x + 4, & x \geq 3 \end{cases}$.

R6

Use the graph to find $\lim_{x\to 2} f(x)$ and $f(2)$.



R7

Evaluate $\lim_{x\to -2} \dfrac{x^3 - 3x + 4}{x^2 + 1}$.

R8

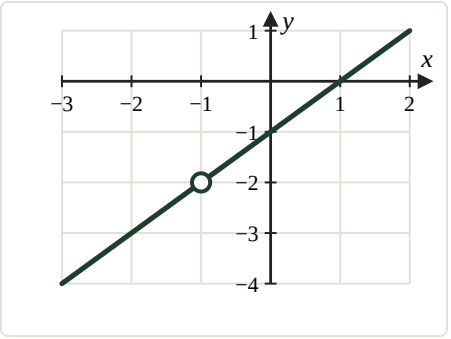
Evaluate $\lim_{x\to -4} \dfrac{x^2 + x - 12}{x + 4}$.

R9

Evaluate $\lim_{x\to 0} \dfrac{\sqrt{25 + x} - 5}{x}$.

R10

Find $\lim_{x\to -1} \dfrac{x^2 - 1}{x + 1}$ graphically, numerically (complete the table), and algebraically.



x	$f(x)$
−1.1	
−1.01	
−0.99	
−0.9	

W1

A ball dropped from a tower falls $s(t) = 16t^2$ feet in t seconds. Its average speed from $t = 1$ to $t = 1 + h$ is $\dfrac{16(1+h)^2 - 16}{h}$. Find the limit as $h \to 0$ and interpret it.

W2

A courier charges $C(w) = 8$ dollars for packages up to and including 2 pounds and $C(w) = 12$ dollars for packages over 2 pounds (up to 5). Find $\lim\limits_{w \to 2^-} C(w)$, $\lim\limits_{w \to 2^+} C(w)$, and $C(2)$. Does $\lim\limits_{w \to 2} C(w)$ exist?

W3

A greenhouse thermometer records the temperature $T(t)$ (°F) near $t = 3$ hours. Estimate $\lim\limits_{t \to 3} T(t)$.

t	$T(t)$
2.9	67.4
2.99	67.94
3.01	68.06
3.1	68.6

W4

A phone plan costs $C(g) = 30$ dollars for $g \le 5$ GB of data and $C(g) = 30 + 10(g - 5)$ dollars for $g > 5$. Find $\lim\limits_{g \to 5} C(g)$.

W5

The area of a square of side s cm is $A(s) = s^2$. The average rate of change of area from $s = 3$ to $s = 3 + h$ is $\dfrac{(3+h)^2 - 9}{h}$. Find the limit as $h \to 0$, with units.

W6

A light sensor's output is modeled by $I(d) = \dfrac{d^2 + 2d - 8}{d - 2}$, which shows an error at $d = 2$ cm. What value do the readings approach as $d \to 2$?

W7

A seedling's height is $H(t) = \sqrt{t + 16}$ cm after t days. Its average growth rate over the first h days is $\dfrac{\sqrt{16 + h} - 4}{h}$. Find the limit as $h \to 0$ and interpret it.

W8

A library fine is \$0.25 for each started day overdue: $F(t) = 0.25\lceil t \rceil$ for $t > 0$ days. Find $\lim\limits_{t \to 3^-} F(t)$ and $\lim\limits_{t \to 3^+} F(t)$.

W9

A bakery's price per loaf $p(t)$ and daily loaves sold $q(t)$ satisfy $\lim\limits_{t \to 6} p(t) = 12$ and $\lim\limits_{t \to 6} q(t) = 150$. Revenue is $R(t) = p(t)q(t)$. Find $\lim\limits_{t \to 6} R(t)$.

W10

A drone's position is $s(t) = t^3$ meters. Its average velocity from $t = 2$ to $t = 2 + h$ is $\dfrac{(2 + h)^3 - 8}{h}$. Find the limit as $h \to 0$.