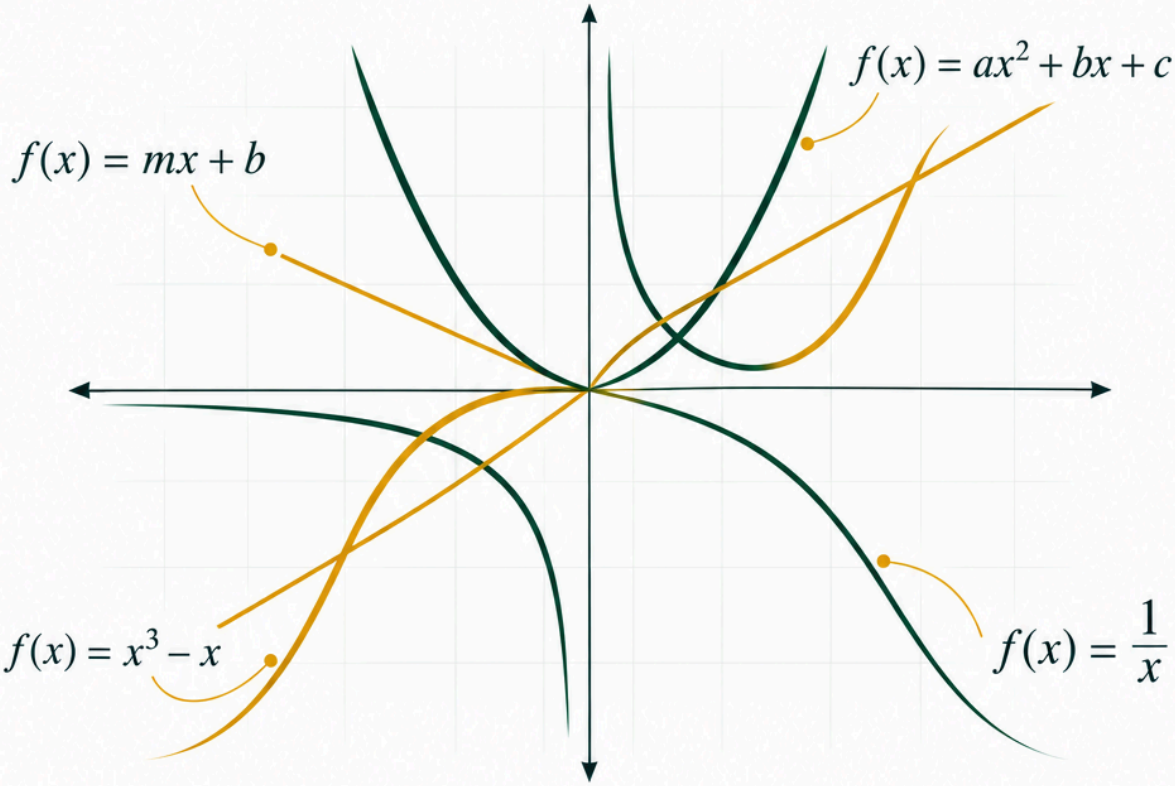
SCHOOL OF MATH

PRE-CALCULUS

CLASS 1

Function Families

Linear • Quadratic • Polynomial • Rational



Recognize patterns. Understand behavior.

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PRE-CALCULUS • CLASS 1 • ITEM 1

What Is a Function?

Each input has exactly one output.

1 INPUT → RULE → OUTPUT

A function takes an input, applies a rule, and produces exactly one output.

Input	Rule	Output
3	$f(x) = 2x + 1$	7

Function notation: $f(3) = 7$.

2 Mapping inputs to outputs

A function assigns each input exactly one output. Different inputs may share an output, but one input cannot have two outputs.

Function (allowed)

$1 \longrightarrow 3$

$2 \longrightarrow 5$

$3 \longrightarrow 7$

Not a function

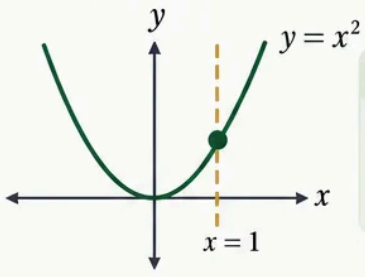
$2 \longrightarrow 4$

$2 \longrightarrow 6$

The input 2 has two outputs, which is not allowed.

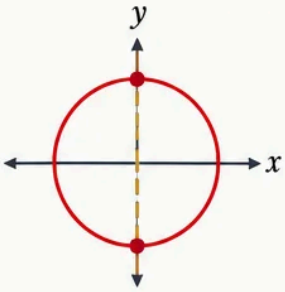
3 The vertical line test

A graph represents y as a function of x when every vertical line intersects it at most once.



Function

The vertical line $x = 1$ intersects the graph once.



Not a function

The vertical line $x = 0$ intersects the graph twice.

One input. Exactly one output.

What is a function? Inputs, rules, and outputs

Q1. Let $f(x) = 3x - 5$. Evaluate $f(4)$, $f(-2)$, and $f(0)$. Write each result in function notation.

Q2. Decide whether each relation is a function. Explain using the input-output rule.

$A:\ 1 \to 4, \quad 2 \to 4, \quad 3 \to 9$
$B:\ \{(1,2),\ (1,5),\ (3,7)\}$

Q3. Let $g(x) = x^2 - 2x$. Find $g(5)$ and $g(-3)$. What do the results tell you about functions and repeated outputs?

Q4. Use the vertical line test to decide which equations define y as a function of x.

$y = |x|, \qquad x = y^2, \qquad x^2 + y^2 = 9, \qquad y = x^3$

Q5. Let $h(x) = 2x + 1$. (a) Find the input x for which $h(x) = 11$. (b) Simplify $h(a + 1)$.

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Domain and Range

Domain = inputs. Range = outputs.

1 Read a set of ordered pairs

The domain is the set of **all first coordinates** (x-values).
The range is the set of all second coordinates (y-values).

$$\{(-2, 1),\ (0, 3),\ (2, 1)\}$$

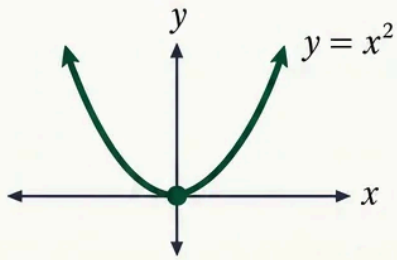
Domain: $\{-2, 0, 2\}$

Range: $\{1, 3\}$

List each distinct value only once.

2 Read a graph

The domain is the set of all x-values shown on the graph.
The range is the set of all y-values shown on the graph.



Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

All real x-values are allowed; outputs are nonnegative.

3 Find restrictions from an equation

The domain consists of all real x-values that make the expression defined.
The range is the set of resulting y-values.

Example 1 $f(x) = \dfrac{1}{x-2}$

$x - 2 \neq 0 \rightarrow x \neq 2$

Domain: $(-\infty, 2) \cup (2, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Example 2 $g(x) = \sqrt{x-3}$

$x - 3 \geq 0 \rightarrow x \geq 3$

Domain: $[3, \infty)$
Range: $[0, \infty)$

Notation tip: [ ] includes an endpoint; ( ) excludes an endpoint. Always use parentheses with infinity.

Ask: Which x-values work? Which y-values occur?

Domain and range

Q6. State the domain and range of the relation. Is it a function?

$$\{(-3, 2),\ (-1, 5),\ (0, 2),\ (4, -1)\}$$

Q7. Find the domain and range of the function. Use interval notation.

$$f(x) = \frac{4}{x + 5}$$

Q8. Find the domain and range of the function. Use interval notation.

$$g(x) = \sqrt{2x - 8}$$

Q9. The graph of $f(x) = -x^2 + 4$ is a parabola that opens downward with vertex $(0, 4)$. State its domain and range from the graph's shape.

Q10. Find the domain of the function. Two restrictions apply.

$$h(x) = \frac{\sqrt{x + 1}}{x - 3}$$

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Linear Functions

A constant rate of change creates a straight line.

1 The linear rule

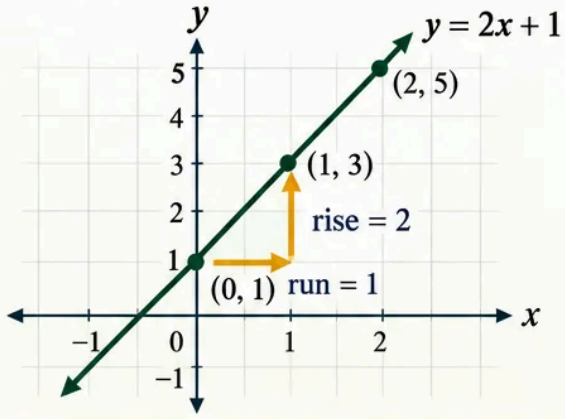
$$f(x) = mx + b$$

m = slope	b = y-intercept
(rate of change)	(initial value)

When x increases by 1, the output changes by m.

2 Slope and intercepts

Example: $f(x) = 2x + 1$



Slope: $m = \dfrac{3-1}{1-0} = 2$

y-intercept: $(0, 1)$

x-intercept: $\left(-\dfrac{1}{2}, 0\right)$

x	-1	0	1	2
$f(x)$	-1	1	3	5

3 Recognize constant change

The slope tells us how the output changes for equal increases in input.

$m > 0$:	**increasing** (line rises left to right)
$m < 0$:	**decreasing** (line falls left to right)
$m = 0$:	**constant** (horizontal line)

Real-world example

A taxi charges \$4 plus \$3 per mile.

$$C(d) = 3d + 4, \quad d \geq 0$$

Slope: \$3 per mile • **Initial cost:** \$4

Takeaway: Equal changes in x produce equal changes in y.

Linear functions: slope, intercepts, and constant change

Q11. For the linear function, state the slope, the y-intercept, and the x-intercept, and say whether the function is increasing or decreasing.

$$f(x) = -3x + 6$$

Q12. A line passes through $(1, 2)$ and $(4, 11)$. Find the slope and write the function in the form $f(x) = mx + b$.

Q13. The table shows outputs of a function at equally spaced inputs. Show the function is linear, write its rule, and find $f(10)$.

x	0	1	2	3
$f(x)$	5	8	11	14

Q14. Classify each linear function as increasing, decreasing, or constant, and give its rate of change.

$$f(x) = 7 - 2x, \qquad g(x) = 5, \qquad h(x) = \frac{x}{2} + 1$$

Q15. A gym charges a $25 enrollment fee plus $40 per month. Write a function $C(m)$ for the total cost after m months, state a sensible domain, find $C(6)$, and find how many months give a total of $345.

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Quadratic Functions

Parabolas, vertices, symmetry, and zeros.

1 Two useful forms

Standard form:

$$f(x) = ax^2 + bx + c$$

$a \neq 0$

Vertex form:

$$f(x) = a(x-h)^2 + k$$

Vertex: (h, k)
Axis of symmetry: $x = h$

$a > 0$: opens upward
$a < 0$: opens downward

2 Read the key features

$$f(x) = x^2 - 4x + 3 = (x-2)^2 - 1$$

Values around the vertex

x	$f(x)$
0	3
1	0
2	−1
3	0
4	3

Axis of symmetry: $x = 2$

Vertex:	$(2, -1)$
Minimum:	-1
Domain:	$(-\infty, \infty)$
Range:	$[-1, \infty)$

3 Find the zeros and intercepts

Set $f(x) = 0$:

$$(x-1)(x-3) = 0$$

$x = 1$ or $x = 3$

x-intercepts: $(1, 0)$ and $(3, 0)$

Set $x = 0$:

$$f(0) = 3$$

y-intercept: $(0, 3)$

Takeaway: The vertex is a minimum when $a > 0$ and a maximum when $a < 0$.

Quadratic functions: vertex, symmetry, and zeros

Q16. Write the function in vertex form. Then state the vertex, axis of symmetry, minimum or maximum value, range, zeros, and y-intercept.

$f(x) = x^2 - 6x + 5$

Q17. The function is given in vertex form. State the vertex, the direction of opening, the maximum or minimum value, the range, the zeros, and the y-intercept.

$g(x) = -2(x + 1)^2 + 8$

Q18. Complete a table of values for $x = -1, 0, 1, 2, 3$. Use the symmetry in the table to identify the axis of symmetry and the vertex.

$f(x) = x^2 - 2x - 3$

x	-1	0	1	2	3
$f(x)$	?	?	?	?	?

Q19. Find the vertex using $x = -\dfrac{b}{2a}$. Then write the function in vertex form and state its range.

$h(x) = 2x^2 + 8x + 3$

Q20. A parabola has zeros at $x = -2$ and $x = 6$ and passes through $(0, -6)$. Write its equation in factored form and standard form, then find its vertex.

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Polynomial Functions

Degree, leading coefficient, zeros, and end behavior.

1 Recognize a polynomial

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$
$$a_n \neq 0$$

Exponents of x are nonnegative integers.

Domain: all real numbers.

Example:

$$f(x) = 2x^3 - 5x^2 + x - 4$$

Degree:	3
Leading coefficient:	2
Constant term:	-4

2 Find zeros from factors

$$p(x) = (x+2)(x-1)^2$$

$x = -2$:
multiplicity 1
→ crosses the x-axis.

$x = 1$:
multiplicity 2
→ touches the x-axis and turns.

A zero r gives an x-intercept $(r, 0)$.

Odd multiplicity: crosses the x-axis.

Even multiplicity: touches the x-axis and turns.

3 Predict end behavior

Degree	Leading coefficient	Left end	Right end
Even	Positive	Up	Up
Even	Negative	Down	Down
Odd	Positive	Down	Up
Odd	Negative	Up	Down

The leading term controls the ends of the graph.

Example: $p(x) = (x+2)(x-1)^2$ has degree 3 and leading coefficient 1: left end down, right end up.

Takeaway: Degree n: at most n real zeros and at most $n-1$ turning points ($n \geq 1$).

Polynomial functions: zeros and end behavior

Q21. State the degree, leading coefficient, and constant term. Describe the end behavior and give the domain.

$$f(x) = -3x^4 + 2x^3 - x + 7$$

Q22. List the zeros with their multiplicities and state whether the graph crosses or touches the x-axis at each. Then give the degree and end behavior.

$$p(x) = (x - 3)(x + 1)^2(x - 5)$$

Q23. Predict the end behavior of each polynomial from its leading term only.

$$q(x) = 2x^5 - x^2 + 4, \qquad r(x) = -x^3 + 6x - 1$$

Q24. Factor completely, find all real zeros, and state the maximum number of turning points and the end behavior.

$$f(x) = x^3 - 4x^2 - 12x$$

Q25. Write a polynomial of least degree with leading coefficient 1 whose only zeros are -2 (multiplicity 2) and 3. Expand it, then state its degree, y-intercept, and end behavior.

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Rational Functions

Domain restrictions, holes, and asymptotes.

1 A ratio of polynomials

$$f(x) = \frac{P(x)}{Q(x)}$$

P and Q are polynomials; $Q(x) \neq 0$.

Exclude every zero of the ORIGINAL denominator from the domain.

Example:

$$g(x) = \frac{1}{x-2}$$

Domain:

$(-\infty, 2) \cup (2, \infty)$

2 Hole or vertical asymptote?

$$f(x) = \frac{(x-1)(x+2)}{(x-1)(x-3)} = \frac{x+2}{x-3}, \quad x \neq 1, 3$$

Canceled factor: $x - 1$

Hole at $x = 1$.

$$y = \frac{1+2}{1-3} = -\frac{3}{2}$$

Hole: $\left(1, -\frac{3}{2}\right)$

Remaining denominator: $x - 3$

Vertical asymptote:

$x = 3$.

Simplifying does *not* restore excluded inputs.

Even though the factor $(x - 1)$ cancels, $x = 1$ is still not in the domain of the original function.

3 Horizontal asymptotes

Compare degrees	Horizontal asymptote
numerator < denominator	$y = 0$
numerator = denominator	$y = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$
numerator > denominator	None

Example: For $f(x) = \frac{x+2}{x-3}$, the horizontal asymptote is $y = 1$.

A graph may cross a horizontal asymptote.

Takeaway: Factor → restrict the domain → simplify → identify holes and asymptotes.

Rational functions: holes and asymptotes

Q26. State the domain, the vertical asymptote, and the horizontal asymptote.

$$g(x) = \frac{3}{x + 4}$$

Q27. Find the domain, any hole (with its coordinates), the vertical asymptote, and the horizontal asymptote.

$$f(x) = \frac{(x - 2)(x + 5)}{(x - 2)(x + 1)}$$

Q28. Find the horizontal asymptote of each function, if one exists. For (a), also give the vertical asymptotes.

(a) $\dfrac{2x^2 + 1}{x^2 - 9}$ (b) $\dfrac{x + 3}{x^2 - 1}$ (c) $\dfrac{x^3 - 1}{x + 2}$

Q29. Factor the numerator and denominator. Then state the domain, the hole, the vertical asymptote, and the horizontal asymptote.

$$h(x) = \frac{x^2 - 9}{x^2 - x - 6}$$

Q30. Find the intercepts and asymptotes of the function. Then decide whether the graph ever crosses its horizontal asymptote.

$$f(x) = \frac{2x + 6}{x - 1}$$

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Function Transformations

Shifts, reflections, stretches, and compressions.

1 Start with a parent function

Linear:	Quadratic:	Cubic:	Reciprocal:
$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$

General transformation rule:

$$g(x) = af(b(x-h)) + k$$

$a \neq 0, \quad b \neq 0.$

2 Read each transformation

Parameter	Effect
h	Shift right h if $h > 0$; left $\lvert h \rvert$ if $h < 0$.
k	Shift up k if $k > 0$; down $\lvert k \rvert$ if $k < 0$.
$\lvert a \rvert$	Vertical scale factor $\lvert a \rvert$.
$a < 0$	Reflect across the x-axis.
$\lvert b \rvert$	Horizontal scale factor $\frac{1}{\lvert b \rvert}$.
$b < 0$	Reflect across the y-axis before shifting.

Scale factor > 1: stretch • Scale factor between 0 and 1: compression.

3 Work through an example

Parent: $f(x) = x^2$

$$g(x) = -2(x-3)^2 + 1$$

Vertex: $(3, 1)$ • Opens downward.

Domain: $(-\infty, \infty)$ • **Range:** $(-\infty, 1]$

Transformations:

1. Shift right 3.
2. Vertical stretch by factor 2.
3. Reflect across the x-axis.
4. Shift up 1.

These changes describe the final graph; apply vertical scaling and reflection before the upward shift.

Takeaway: Inside changes affect x; outside changes affect y.

Function transformations: shifts, reflections, and scaling

Q31. Describe how the graph of g is obtained from the parent $f(x) = x^2$. State the new vertex and the range.

$$g(x) = (x + 4)^2 - 3$$

Q32. List the transformations, in order, that take the parent $f(x) = x^3$ to g. Where does the parent point $(0, 0)$ end up, and where does $(1, 1)$ end up?

$$g(x) = -(x - 2)^3 + 5$$

Q33. The function g is a transformation of the reciprocal parent $f(x) = \dfrac{1}{x}$. Describe the transformations and state the new asymptotes, domain, and range.

$$g(x) = \frac{1}{x - 3} + 2$$

Q34. Write the equation of the function obtained from $f(x) = x^2$ by reflecting across the x-axis, stretching vertically by a factor of 3, shifting left 1, and shifting up 4. Give the vertex and range.

Q35. Compare $g(x) = \left(\frac{1}{2}x\right)^3$ with the parent $f(x) = x^3$. Describe the horizontal transformation, find where the parent point $(2, 8)$ lands, and show that the same graph can be described as a vertical change.

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Compare Function Families

Connect equations, tables, and real-world models.

1 Recognize the structure

Family	Example	Key clue
Linear	$f(x) = 2x + 1$	Constant rate of change
Quadratic	$f(x) = x^2$	Degree 2; a parabola
Polynomial	$f(x) = x^3 - x$	Nonnegative integer powers
Rational	$f(x) = \frac{1}{x}$	A ratio of polynomials

Linear and quadratic functions are polynomials.
Every polynomial is also rational: $P(x) = \frac{P(x)}{1}$.

2 Compare tables

x	$2x+1$	x^2	x^3	$\frac{1}{x}$
1	3	1	1	1
2	5	4	8	$\frac{1}{2}$
3	7	9	27	$\frac{1}{3}$
4	9	16	64	$\frac{1}{4}$

- **Linear outputs:** first differences are constant (+2).
- **Quadratic outputs:** second differences are constant (+2).

These patterns use equally spaced x-values; a finite table alone does not prove the model.

3 Choose a model that fits

Linear
Taxi fare:
$C(d) = 3d + 4$
$d \geq 0$

Quadratic
Square area:
$A(s) = s^2$
$s \geq 0$

Cubic polynomial
Cube volume:
$V(s) = s^3$
$s \geq 0$

Rational
Travel time for 120 miles:
$T(v) = \frac{120}{v}$
$v > 0$

d: miles; s: side length; v: miles per hour; T: hours.

Takeaway: Choose the model from the relationship, then check its domain and units.

Compare function families

Q36. Classify each function as linear, quadratic, polynomial (give the degree), or rational. Note any function that belongs to more than one family.

$$f(x) = 5 - 2x, \qquad g(x) = x^3 - 2x^2 + 1, \qquad h(x) = \frac{x+1}{x^2-4}, \qquad k(x) = 3x^2 - x$$

Q37. Use first and second differences to identify the family suggested by the table. Then find a rule of the form $y = x^2 + c$ that fits every row.

x	1	2	3	4
y	4	7	12	19

Q38. Decide whether the table is better described by a linear, quadratic, or rational model, and write the rule.

x	1	2	3	4
y	12	6	4	3

Q39. Choose a function family for each situation, write the model, and state a sensible domain.

Q40. Compare $f(x) = 10x + 5$ (linear) and $g(x) = x^2$ (quadratic) for positive integers x. Find the smallest integer x for which $g(x) > f(x)$, and explain what this says about the two families in the long run.

Homework review

R1. Let $f(x) = x^2 + 3x$. Find $f(2)$, $f(-4)$, and $f(a)$. Then decide whether $\{(2,5),(3,5),(4,7)\}$ is a function.

R2. Find the domain and range in interval notation.

$$f(x) = \sqrt{6 - 2x}$$

R3. A line passes through $(-2, 7)$ and $(3, -8)$. Find the slope, write the linear function, and find both intercepts.

R4. Find the vertex, maximum or minimum value, zeros, y-intercept, and range.

$$f(x) = -x^2 + 8x - 12$$

R5. State the degree, leading coefficient, zeros with multiplicity and behavior, end behavior, and y-intercept.

$$p(x) = -2(x + 1)(x - 2)^2(x - 4)$$

Homework review (continued)

R6. Find the domain, hole, vertical asymptote, and horizontal asymptote.

$$f(x) = \frac{x^2 - 4}{x^2 + x - 6}$$

R7. Describe the transformations from $f(x) = x^2$ to g, then give the vertex and range.

$$g(x) = \tfrac{1}{2}(x - 5)^2 - 2$$

R8. Two tables use the same inputs. Identify the family each table suggests and write a rule for each.

x	0	1	2	3
Table A: y	2	5	8	11
Table B: y	1	2	5	10

Homework word problems

W1. A plumber charges a \$60 call-out fee plus \$45 per hour. Write the cost function $C(h)$, find the cost of a 3-hour job, and find how many hours produce a bill of \$285.

W2. A tank holds 500 gallons and drains at a constant 20 gallons per minute. Write $V(t)$ for the volume after t minutes, give the domain in context, and find when the tank is empty.

W3. A trip is 240 miles. Write the travel time $T(v)$ in hours as a function of average speed v in miles per hour. Find $T(60)$ and $T(80)$, state the domain, and explain the meaning of the horizontal asymptote.

W4. A ball is thrown upward from a height of 5 feet. Its height in feet after t seconds is modeled below. Find the maximum height and when it occurs, the starting height, and the time the ball reaches the ground (to the nearest hundredth).

$$h(t) = -16t^2 + 48t + 5$$

W5. A theater sells $200 - 5p$ tickets when the price is p dollars. Write the revenue function $R(p)$, give the domain in context, and find the price that maximizes revenue and the maximum revenue.

Homework word problems (continued)

W6. A rectangular garden is enclosed with 60 meters of fencing. Let w be the width. Write the area $A(w)$, state the domain, and find the dimensions that give the largest area.

W7. An open box is made from a 12-inch square of cardboard by cutting an x-inch square from each corner and folding up the sides. Write the volume $V(x)$ in factored and expanded form, state the degree and domain, and find $V(2)$ and $V(3)$.

W8. The water level in a wave tank, in centimeters above the rest level, is modeled for $0 \le t \le 5$ seconds by the function below. Factor it, find its zeros and their multiplicities, interpret what happens at each zero, and find $h(1)$ and $h(5)$.

$$h(t) = t^3 - 6t^2 + 9t$$

W9. A workshop spends \$2,000 on setup plus \$50 per unit to produce x units. The average cost per unit is given below. State the domain, find the average cost for 100 and 400 units, find how many units give an average cost of \$60, and interpret the horizontal asymptote.

$$\bar{C}(x) = \frac{50x + 2000}{x}$$

W10. The concentration of a medicine in the bloodstream, in milligrams per liter, t hours after a dose is modeled below. Find $C(0)$, $C(2)$, and $C(6)$; state the horizontal asymptote and its meaning; and show, without calculus, that the concentration never exceeds 1 mg/L.

$$C(t) = \frac{4t}{t^2 + 4}, \qquad t \ge 0$$

Homework word problems (continued)

W11. A lemonade stand's daily profit in dollars depends on the price p per cup. Identify the transformations of the parent p^2, state the best price and maximum profit, find the prices that give zero profit, and give the domain over which the profit is nonnegative.

$$P(p) = -2(p - 3)^2 + 18$$

W12. A lab records a bacteria count N (in thousands) at equally spaced hours. Use differences to choose a family, write a rule in the form $N(t) = t^2 + c$, predict $N(10)$, find when the count reaches 84 thousand, and explain why the table alone does not prove the model.

t (hours)	0	1	2	3	4
N (thousands)	3	4	7	12	19