Linear Geometry

Creating, graphing, and measuring lines and line segments

1.1 Solve the equation: $3x-(x-7)=4x-5$.

Distribute –1 through the parentheses and combine like terms.

$$3x-x+7=4x-5$$
$$2x+7=4x-5$$

Subtract $4x$ and 7 from both sides of the equation to separate the variable and constant terms.

$$\begin{array}{rcrcrcr} 2x & + & 7 & = & 4x & - & 5 \\ -4x & - & 7 & & -4x & - & 7 \\ \hline -2x & & & = & & - & 12 \end{array}$$

Divide both sides by –2 to get the solution.

$$\frac{-2x}{-2}=\frac{-12}{-2}$$
$$x=6$$

1.2 Calculate the slope, m, of the line $4x-3y=9$.

Slope-intercept form of a line is $y = mx + b$, where m is the slope of the line and b is the y-intercept.

Solve the equation for y in order to rewrite it in slope-intercept form.

$$-3y=-4x+9$$
$$y=\frac{4}{3}x-3$$

The slope of the line is the coefficient of x: $m=\frac{4}{3}$.

1.3 Prove that the slope of a line in standard form, $Ax+By=C$, is $-\frac{A}{B}$.

Write the equation in slope-intercept form by solving it for y.

$$Ax+By=C$$
$$By=-Ax+C$$
$$y=-\frac{A}{B}x+\frac{C}{B}$$

The coefficient of x is the slope of the line: $m=-\frac{A}{B}$.

The equation's in standard form if it has: (1) No fractions, (2) Only x- and y-terms on the left side, (3) Just the constant on the right side, and (4) A positive x-coefficient.

1.4 Rewrite the linear equation $3x-4\left(x-\frac{2}{3}y\right)=\frac{4}{5}x-(7y+3)$ in standard form.

Distribute the constants and combine like terms.

$$3x-4x+\frac{8}{3}y=\frac{4}{5}x-7y-3$$
$$-x+\frac{8}{3}y=\frac{4}{5}x-7y-3$$

Multiply by 15, the least common denominator, to eliminate fractions.

$$-15x + 40y = 12x - 105y - 45$$

Separate the variable and constant terms.

$$-27x + 145y = -45$$
$$27x - 145y = 45$$

Multiply the entire equation by −1 so that the x-coefficient is positive. (It's a requirement of standard form.)

1.5 Write the equation of the line passing through the points (–3,–8) and (–6,2) in slope-intercept form.

Calculate the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-8)}{-6-(-3)} = \frac{10}{-3} = -\frac{10}{3}$$

Think of the point (–3,–8) as (x_1,y_1) and (–6,2) as (x_2,y_2), so $x_1 = -3$, $y_1 = -8$, $x_2 = -6$, and $y_2 = 2$.

Substitute the slope into the slope-intercept formula ($y = mx + b$) for m, replace x and y using one of the coordinate pairs, and solve for b.

$$y = mx + b$$
$$-8 = -\frac{10}{3}(-3) + b$$
$$-8 = 10 + b$$
$$b = -18$$

Substitute m and b into the slope-intercept formula.

$$y = mx + b$$
$$y = -\frac{10}{3}x - 18$$

1.6 Calculate the x- and y-intercepts of $3x - 4y = -6$ and use them to graph the line.

To calculate the x-intercept, substitute 0 for y and solve for x. Similarly, substitute 0 for x to calculate the y-intercept.

$$3(0) - 4y = -6 \qquad 3x - 4(0) = -6$$
$$-4y = -6 \qquad 3x = -6$$
$$y = \frac{3}{2} \qquad x = -2$$

Therefore, the graph of $3x - 4y = -6$ intersects the x-axis at (–2,0) and the y-axis at $\left(0,\frac{3}{2}\right)$, as illustrated by Figure 1-1.

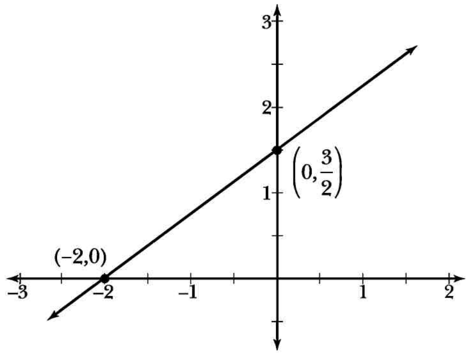

Figure 1-1

The graph of $3x - 4y = -6$ with its x- and y-intercepts identified.

The point-slope formula creates an equation based on the slope of the line, m, and a point on the line, (x_1,y_1). Don't plug anything in for the x and y that don't have little numbers next to them.

1.7 Assume that line p contains the point (–3,1) and is parallel to $x - 4y = 1$. Write the equation of p in slope-intercept form.

Calculate the slope of $x - 4y = 1$ using the method of Problem 1.3.

$$m = -\frac{A}{B} = -\frac{1}{-4} = \frac{1}{4}$$

Plug this slope and the coordinates $(x_1,y_1) = (-3,1)$ into the point-slope formula.

$$y - y_1 = m(x - x_1)$$
$$y - 1 = \frac{1}{4}(x - (-3))$$
$$y - 1 = \frac{1}{4}x + \frac{3}{4}$$

Solve for y to express the equation in slope-intercept form.

$$y = \frac{1}{4}x + \frac{7}{4}$$

Note: Problems 1.8 - 1.10 refer to parallelogram ABCD in Figure 1-2.

1.8. According to a basic Euclidean geometry theorem, the diagonals of a parallelogram bisect each other. Demonstrate this theorem for parallelogram *ABCD*.

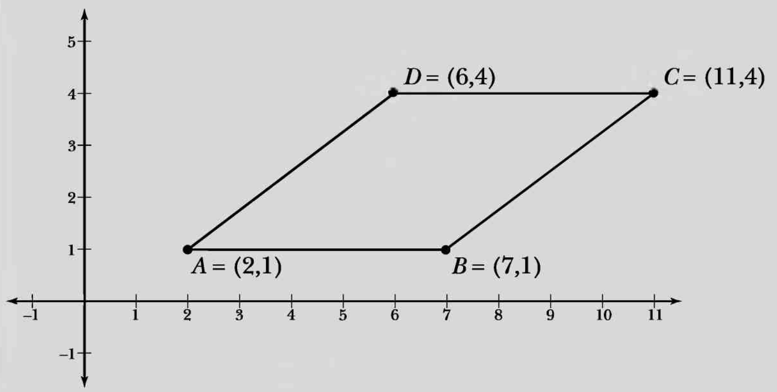

Figure 1-2

Parallelogram ABCD.

The midpoint of a line segment with endpoints (x_1,y_1) and (x_2,y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. In other words, the x-coordinate of a segment's midpoint is the average of the x-coordinates of its endpoints. The y-coordinate works the same way.

Calculate the midpoints of $\overline{AC}$ and $\overline{BD}$; the diagonals bisect one another if and only if those midpoints are equal.

Midpoint of $\overline{AC}$:

$$\left(\frac{2+11}{2},\frac{1+4}{2}\right)=\left(\frac{13}{2},\frac{5}{2}\right)$$

Midpoint of $\overline{BD}$:

$$\left(\frac{7+6}{2},\frac{1+4}{2}\right)=\left(\frac{13}{2},\frac{5}{2}\right)$$

Note: Problems 1.8 - 1.10 refer to parallelogram ABCD in Figure 1-2.

1.9 Prove that *ABCD* is a rhombus by verifying that its sides are congruent.

Apply the distance formula four times, once for each side.

The distance between the points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

$$AB=\sqrt{(7-2)^2+(1-1)^2}=\sqrt{25+0}=5 \qquad BC=\sqrt{(11-7)^2+(4-1)^2}=\sqrt{16+9}=5$$

$$CD=\sqrt{(6-11)^2+(4-4)^2}=\sqrt{25+0}=5 \qquad AD=\sqrt{(6-2)^2+(4-1)^2}=\sqrt{16+9}=5$$

Note: Problems 1.8 - 1.10 refer to parallelogram ABCD in Figure 1-2.

1.10 Prove that *ABCD* is a rhombus by verifying that its diagonals are perpendicular to one another.

Calculate the slopes of the diagonals using the slope formula from Problem 1.5.

$$\text{Slope of } \overline{AC}\text{: } m_1=\frac{4-1}{11-2},\quad m_1=\frac{1}{3} \qquad \text{Slope of } \overline{BD}\text{: } m_2=\frac{4-1}{6-7},\quad m_2=\frac{3}{-1}$$

The diagonals are negative reciprocals, so the line segments are perpendicular.

~~Parallel~~ Parallel lines have equal slopes. The slopes of perpendicular lines are reciprocals of one another and have opposite signs.

Linear Inequalities and Interval Notation

Goodbye equal sign, hello parentheses and brackets

1.11 Write the expression $x \ge -4$ using interval notation.

An interval is defined by the two values that bound an inequality statement, the lower followed by the upper bound. You must indicate whether or not each endpoint is included in the interval. (A bracket next to an endpoint signifies inclusion, and a parenthesis indicates exclusion.)

Always use parentheses next to ∞ when writing intervals. You can't "include" something that's not a finite, real number.

Any number greater than or equal to −4 makes this statement true; −4 is the lower bound and must be included. The upper bound is infinity. Therefore, $x \ge -4$ is written $[-4,\infty)$.

1.12 Write the expression $x < 10$ using interval notation.

The upper bound is 10 and should be excluded (since 10 is not less than 10). Any number less than 10 makes this statement true; there are infinitely many such values in the negative direction, so the lower bound is $-\infty$. Therefore, the inequality statement is written $(-\infty,10)$.

RULE OF THUMB: Use a bracket if the inequality symbol next to the number is $\le$ or $\ge$—otherwise use a parenthesis. Always use parentheses next to ∞ and $-\infty$.

1.13 Write the expression $6 \ge x > -1$ using interval notation.

The lower bound must always precede the upper bound, regardless of how the expression is written: $(-1,6]$.

1.14 Write the solution to the inequality using interval notation: $4x - 2 > x + 13$.

Separate variables and constants, then divide by the coefficient of x.

$$4x - x > 13 + 2$$
$$3x > 15$$
$$x > 5$$

Write the solution in interval notation: $(5,\infty)$.

1.15 Write the solution to the inequality using interval notation: $3(2x - 1) - 5 \le 10x + 19$.

Distribute the constant, combine like terms, and isolate x on the left side of the equation.

$$6x - 3 - 5 \le 10x + 19$$
$$6x - 8 \le 10x + 19$$
$$-4x \le 27$$

If you multiply or divide both sides of an inequality by a negative number, reverse the inequality sign. In this case, $\le$ becomes $\ge$.

Dividing by a negative constant fundamentally changes the inequality: $x \ge -\frac{27}{4}$.

Write the solution in interval notation: $\left[-\frac{27}{4},\infty\right)$.

1.16 Graph the inequality: $-2 \le x < 3$.

Rewrite the inequality as an interval: $[-2,3)$. To graph the interval on a number line, place a dot at each boundary (closed dots for included boundaries and open dots for excluded boundaries). All values between those boundaries belong to the interval, so darken the number line between the dots, as illustrated by Figure 1-3.

Some textbooks use brackets instead of closed dots and parentheses instead of open dots on the number line.

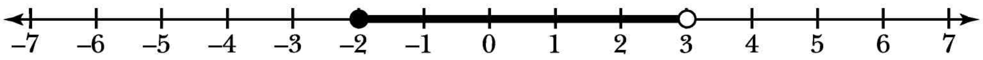

Figure 1-3 *The graph of $-2 \le x < 3$ includes the interval boundary $x = -2$, but excludes $x = 3$.*

1.17 Graph the inequality: $x > -1$.

There is no upper bound for the interval $(-1,\infty)$, but all values greater than -1 satisfy the inequality. Therefore, shade all numbers greater than -1 on the number line, as illustrated in Figure 1-4.

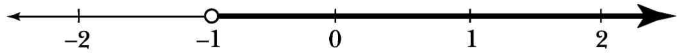

Figure 1-4 *The graph of $x > -1$ excludes the lower boundary, $x = -1$.*

1.18 Solve and graph the inequality: $-7 \le 1 - 2x < 11$.

Isolate $-2x$ in the middle of the compound inequality by subtracting 1 from each expression. Next, divide each expression by -2 to isolate x, reversing the inequality signs as you do.

$$-7-1 \le -2x < 11-1$$
$$\frac{-8}{-2} \ge \frac{-2x}{-2} > \frac{10}{-2}$$
$$4 \ge x > -5$$

The graph of the solution, $(-5,4]$, is illustrated in Figure 1-5.

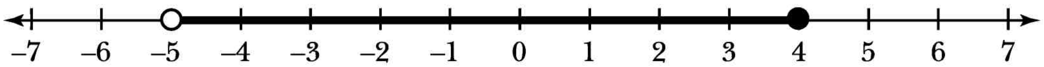

Figure 1-5 *The graph of $-7 \le 1 - 2x < 11$ includes $x = 4$ and excludes $x = -5$.*

1.19 Graph the inequality: $y < -\frac{1}{3}x + 2$.

This inequality contains two variables, x and y, so it must be graphed on the coordinate plane. Note that the inequality is solved for y and (apart from the inequality sign) looks like a linear equation in slope-intercept form. The linear inequality has y-intercept $(0,2)$ and slope $-\frac{1}{3}$.

Stick the negative sign on top: $\frac{-1}{3}$. Starting at the y-intercept, go down 1 unit, right 3 units, and mark the point. Connect the dots to graph the line.

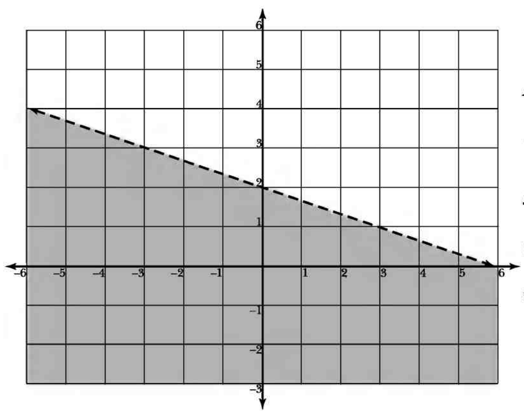

Figure 1-6

The graph of $y < -\frac{1}{3}x + 2$ is dotted rather than solid, because it is excluded from the solution (like an open dot indicates exclusion from an inequality graph on a number line).

By shading the region, you're saying, "All of these points, not just the one I tested, make the inequality true."

The dotted graph separates the coordinate plane into two regions (one above and one below the line). To determine which region represents the solution, choose a point (x,y) from one of the regions and substitute the values into the inequality. If the resulting statement is true, shade the region that contains that point. If not, shade the other region.

1.20 Solve the equation: $2x - y \le 4$.

Solve the inequality for y.

$$-y \le -2x + 4$$
$$y \ge 2x - 4$$

If you don't feel like testing points to figure out where to shade, solve the equation for y and use this rule of thumb: Shade above the line for "greater than" and below the line for "less than."

This graph is solid (not dotted) because the line itself belongs to the solution. Shade the region above the line, as illustrated in Figure 1-7.

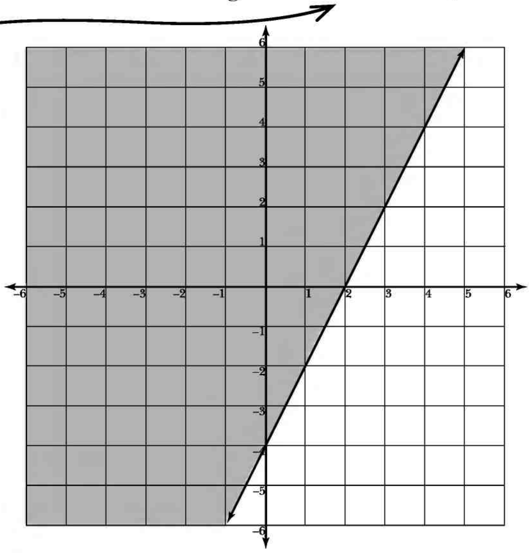

Figure 1-7

All of the ordered pairs above the line are valid solutions to the inequality $2x - y \le 4$.

Absolute Value Equations and Inequalities

Solve two things for the price of one

1.21 Solve the equation: $|3x - 7| = 8$.

In order for this statement to be valid, the absolute value expression must either equal 8 (since $|8| = 8$) or –8 (since $|-8| = 8$).

You're not asked to graph the solution because a number line with dots at $-\frac{1}{3}$ and at 5 isn't very interesting.

$$3x - 7 = 8 \qquad 3x - 7 = -8$$
$$3x = 15 \qquad 3x = -1$$
$$x = 5 \qquad x = -\frac{1}{3}$$

The solution is $x = -\frac{1}{3}$ or $x = 5$.

1.22 Solve the equation: $1 - 2|x + 6| = -4$.

Isolate the absolute value expression on the left side of the equation.

$$-2|x+6| = -5$$

$$|x+6| = \frac{5}{2}$$

Apply the technique described in Problem 1.21.

Remove the absolute value bars and create two equations—one with a positive right side and one with a negative right side.

$$x+6 = \frac{5}{2} \qquad\qquad x+6 = -\frac{5}{2}$$

$$x = \frac{5}{2} - \frac{12}{2} \qquad\qquad x = -\frac{5}{2} - \frac{12}{2}$$

$$x = -\frac{7}{2} \qquad\qquad x = -\frac{17}{2}$$

1.23 Solve the equation: $9 - 3|x+2| = 15$.

Isolate the absolute value expression on the left side of the equation.

$$-3|x+2| = 6$$

$$|x+2| = -2$$

This equation has no solution.

Absolute values always produce a positive number, so there's no way something in absolute values can equal -2.

1.24 Solve the inequality: $|x-5| < 1$.

The solution to the absolute value inequality $|x+a| < b$, where a and b are real numbers (and $b > 0$), is equivalent to the solution of the compound inequality $-b < x + a < b$.

$$-1 < x-5 < 1$$

Drop the inequality bars, stick a matching inequality sign on the left, and then put the opposite of the right side on the left side.

Solve the inequality using the method described in Problem 1.18.

$$-1+5 < x < 1+5$$

$$4 < x < 6$$

The solution, in interval notation, is (4,6).

1.25 Graph the solution to the inequality: $2|x-7| - 5 \le -1$.

Isolate the absolute value expression on the left side of the inequality.

$$2|x-7| \le -1+5$$

$$\frac{2|x-7|}{2} \le \frac{4}{2}$$

$$|x-7| \le 2$$

Create a compound inequality (as explained in Problem 1.24) and solve.

$$\begin{aligned} -2 &\le x-7 \le 2 \\ -2+7 &\le x \le 2+7 \\ 5 &\le x \le 9 \end{aligned}$$

The solution, [5,9], is graphed in Figure 1-8.

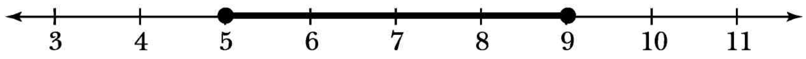

Figure 1-8 *The solution graph of $2|x-7|-5\le -1$ is a closed interval because both endpoints ($x = 5$ and $x = 9$) are included.*

1.26 Solve the inequality: $|2x+5|\ge 3$.

Rewrite an inequality of form $|ax+b|\ge c$ as two new inequalities, $ax+b\ge c$ and $ax+b\le -c$, and solve. The union of the solutions is equivalent to the solution of the original inequality.

$$\begin{array}{rcr} 2x+5\ge 3 & & 2x+5\le -3 \\ 2x\ge -2 & \text{or} & 2x\le -8 \\ x\ge -1 & & x\le -4 \end{array}$$

The solution, in interval notation, is $(\infty,-4]$ or $[-1,\infty)$. The word "or" does not imply that either interval by itself is an acceptable answer, but rather that both intervals together (and therefore all values from both intervals) constitute the solution.

To solve a > or ≥ absolute value inequality, set up two inequalities without absolute value bars. One will match the original inequality. The other looks the same on the left side, but the number on the right is negative and the inequality sign is reversed.

1.27 Solve the inequality and graph the solution: $2-3|x+1|<-5$.

Isolate the absolute value expression on the left side of the inequality.

$$-3|x+1|<-7$$
$$|x+1|>\frac{7}{3}$$

Dividing by –3 reverses the inequality sign; apply the solution method outlined in Problem 1.26.

$$\begin{array}{rcr} x+1>\dfrac{7}{3} & & x+1<-\dfrac{7}{3} \\ & \text{or} & \\ x>\dfrac{4}{3} & & x<-\dfrac{10}{3} \end{array}$$

The solution is $\left(-\infty,-\frac{10}{3}\right)$ or $\left(\frac{4}{3},\infty\right)$. Graph both intervals on the same number line to generate the graph of $2-3|x+1|<-5$, as illustrated by Figure 1-9.

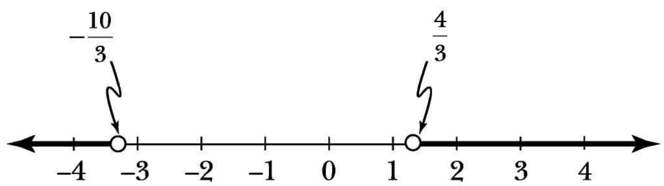

Figure 1-9 *All real numbers satisfy the inequality $2-3|x+1|<-5$, except those on the interval $\left[-\frac{10}{3},\frac{4}{3}\right]$.*

Systems of Equations and Inequalities

Find a common solution shared by multiple equations or inequalities

1.28 Solve the following system of equations using the substitution method.

$$\begin{cases} -8x+2y=-5 \\ 2x-y=1 \end{cases}$$

Solve the second equation for y and substitute its value into the first equation.

Solving $2x - y = 1$ for y gives you $y = 2x - 1$. This allows you to replace y in the other equation with $2x - 1$.

$$\begin{aligned} -8x+2(2x-1)&=-5 \\ -8x+4x-2&=-5 \\ -4x&=-3 \\ x&=\frac{3}{4} \end{aligned}$$

Substitute this x-value into the equation solved for y at the start of the problem.

$$y=2x-1=2\left(\frac{3}{4}\right)-1=\frac{6}{4}-1=\frac{1}{2}$$

The coordinate pair (x,y) is the solution: $\left(\frac{3}{4},\frac{1}{2}\right)$.

$\left(\frac{3}{4},\frac{1}{2}\right)$ is also the point where the graphs of the lines $-8x + 2y = -5$ and $2x - y = 1$ intersect.

1.29 Solve the following system of equations using the elimination method.

$$\begin{cases} 2x-5y=-11 \\ 3x+13y=4 \end{cases}$$

To eliminate x from the system, multiply the first equation by –3, multiply the second equation by 2, and then add the equations together.

Another option is to eliminate y by multiplying the top equation by 13 and the bottom equation by 5.

$$\begin{array}{rrcr} -6x & +15y & = & 33 \\ 6x & +26y & = & 8 \\ \hline & 41y & = & 41 \\ & y & = & 1 \end{array}$$

Substitute $y = 1$ into either of the original equations and solve for x.

$$\begin{aligned} 2x-5y&=-11 \\ 2x-5(1)&=-11 \\ 2x&=-11+5 \\ x&=-\frac{6}{2}=-3 \end{aligned}$$

The point (–3,1) is the solution to the system of equations.

1.30 Solve the system of equations: $\begin{cases} x-6y=24 \\ \frac{1}{3}x-2y=8 \end{cases}$

It's "easily solved for x" because the coefficient of x is 1. Not needing to divide by an x-coefficient means one less fraction to deal with.

Use the substitution technique, as the first equation is easily solved for x: $x = 6y + 24$.

$$\frac{1}{3}(6y+24)-2y=8$$
$$2y+8-2y=8$$
$$8=8$$

The end result is a true statement ($8 = 8$), but no variables remain. This indicates that the equations of the system are multiples of each other (dividing the first equation by 2 results in the second equation); the system is therefore *dependent* and possesses infinitely many solutions.

Even though the equations in "dependent" systems look different, their graphs are exactly the same. No matter what x you plug into the equations, you'll get the same y, so they intersect at infinitely many points and have infinitely many common solutions.

1.31 Determine the real number value of k in the system of equations below that makes the system indeterminate.

$$\begin{cases} x-6y=-13 \\ 4x-ky=1 \end{cases}$$

An indeterminate system of equations has no solution. Consider this geometric explanation: If the solution to a system of equations is the point(s) at which the graphs of its equations intersect, then an indeterminate system has no solution because the graphs of the linear equations do not intersect. The slope of the first line is $-\frac{1}{-6}=\frac{1}{6}$, and the slope of the second line is $-\frac{4}{-k}=\frac{4}{k}$. Set the slopes equal to create parallel lines, and solve for k.

This is the slope shortcut formula from problem 1.3.

$$\frac{4}{k}=\frac{1}{6}$$

Cross multiply to solve the proportion.

$$4\cdot 6=k\cdot 1$$
$$24=k$$

1.32 Graph the solution to the below system of inequalities.

$$\begin{cases} y<3 \\ x\geq -4 \\ y>\frac{2}{3}x-1 \end{cases}$$

Graph the inequalities on the same coordinate plane, as illustrated by Figure 1-10. The region of the plane upon which the shaded solutions of all three inequality graphs overlap is the solution to the system.

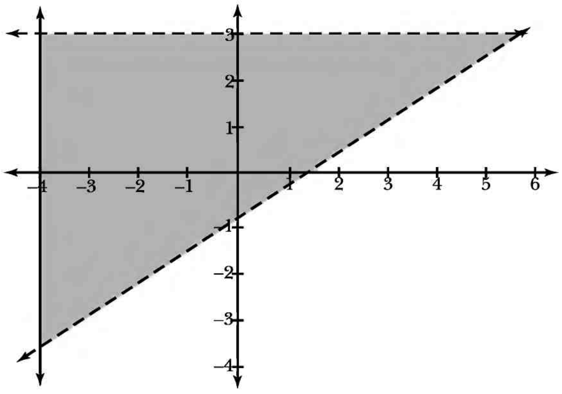

Figure 1-10

The solution to a system of inequalities is a two-dimensional shaded region. Note that $y = 3$ is a horizontal line 3 units above the x-axis and $x = -4$ is a vertical line four units left of the y-axis.

1.33 Solve the following system of equations.

$$\begin{cases} 3x + 2y - z = 0 \\ 5x - y - 8z = 9 \\ x + 4y - 3z = -22 \end{cases}$$

Solve the first equation for z: $z = 3x + 2y$. Use this expression to replace z in the other two equations of the system.

$$\begin{aligned} 5x - y - 8z &= 9 \\ 5x - y - 8(3x + 2y) &= 9 \\ 5x - y - 24x - 16y &= 9 \\ -19x - 17y &= 9 \end{aligned}$$

$$\begin{aligned} &x + 4y - 3z = -22 \\ &x + 4y - 3(3x + 2y) = -22 \\ &x + 4y - 9x - 6y = -22 \\ &-8x - 2y = -22 \\ &-4x - y = -11 \end{aligned}$$

You can divide $-8x - 2y = -22$ by any number except 0 without affecting the solution. To make things a little easier, you should divide everything in this equation by 2.

You're left with a system of two equations in two variables:

$$\begin{cases} -19x - 17y = 9 \\ \quad -4x - y = -11 \end{cases}$$

Solve this system using substitution. (Solve the second equation for y to get $y = -4x + 11$ and substitute that expression into the other equation.)

$$\begin{aligned} -19x - 17(-4x + 11) &= 9 \\ -19x + 68x - 187 &= 9 \\ 49x &= 196 \\ x &= 4 \end{aligned}$$

Plug $x = 4$ into the equation you previously solved for y.

$$\begin{aligned} y &= -4x + 11 \\ y &= -4(4) + 11 \\ y &= -5 \end{aligned}$$

Plug $x = 4$ and $y = -5$ into the equation previously solved for z.

$$\begin{aligned} z &= 3x + 2y \\ z &= 3(4) + 2(-5) \\ z &= 2 \end{aligned}$$

The solution to the system of equations is $(x,y,z) = (4,-5,2)$.